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Random convection

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The main thrust of this work is to treat the initial convective phase of a fluid heated from below as a statistical initial value problem. The advantage of the approach is that it allows a continuous bandwidth of modes to be represented in the initial spectrum. We show that if the initial disturbance field is small and has a sufficiently smooth spectrum, then a natural statistical selection process chooses from the initial disorder a perfectly ordered field of single rolls. The scale of this roll is the scale corresponding to the most critical wave-number obtained from the linear stability problem. We relate this solution to the optimal solution which would be obtained by the upper bound procedures of Howard, Malkus and Busse. Moreover, we show in addition, that if the initial disturbance field is weighted in favour of a particular single roll whose scale is close to critical, the final solution reflects the initial condition providing a certain stability criterion is met. In the two-dimensional case we analyze, this turns out to be the Eckhaus stability condition previously obtained by a discrete multimodal analysis.

1. Introduction

One of the intriguing features of mathematical physics is the evolution of order from initial disorder. A classical example is the success of macroscopic gas dynamics based upon the premise that the microscopic molecular disorder relaxes very quickly to statistical order, a state of thermodynamic equilibrium. A further example of this phenomenon is the way in which a field of weakly coupled dispersive random waves, initially non-Gaussian, can relax to a state *close enough* to Gaussianity so as to permit a finite closure on the hierarchy of moment equations (Benney & Newell 1969).

In this work we discuss a related phenomenon, namely the evolution of macroscopic order out of initial macroscopic disorder. The problem we address is that of thermal convection in a horizontally infinite layer heated from below. It is known that when the temperature difference (in dimensionless units, the Rayleigh number Ra) exceeds a certain critical value, the purely conductive solution is unstable and convective motion of cellular structure and with a particular cell size sets in. Malkus & Veronis (1958) discussed the finite amplitude nature of the steady convective motions whose cell size corresponded to the most critical wave-number. However, it is clear from the stability diagram (obtained from a linear stability analysis of the purely conductive solution, see figure 1) that a continuous finite bandwidth of modes is possible.

If we define L to be the difference between k the wave-number of the motion and k_c the wave-number of the most critical mode, then in the case when both boundaries have free surface boundary conditions (and so $Ra_c = \frac{27}{4}\pi^4$, $k_c = \pi/\sqrt{2}$;) the continuous range of unimodal solutions correspond to the band,

$$L^2 < \frac{3\pi^2}{8}\frac{Ra - Ra_c}{Ra_c}. \tag{1.1}$$

Schlüter, Lortz & Busse (1965) examined the stability of such solutions (corresponding to wave vector $(k_c+L, 0)$). They concluded that by virtue of three-

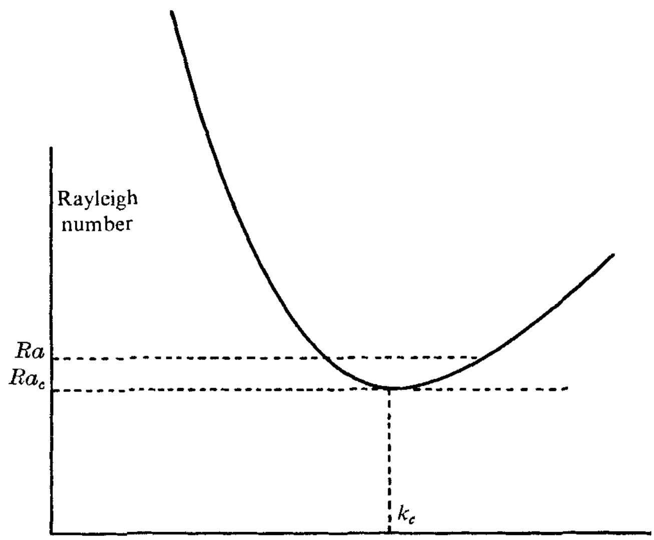

FIGURE 1. Rayleigh number *vs.* wave-number diagram separating regions of stability, instability of linear conductive profile.

dimensional $(0, k_c)$ and oblique $(k_c - L, \pm(\sqrt{2}\pi L)^{\frac{1}{2}})$ mode instabilities the band of solutions, which are stable to infinitesimal perturbations about their finite amplitude steady states, is restricted to the range,

$$0 < L < \gamma^2\left(\frac{3\pi^2}{8}\frac{Ra - Ra_c}{Ra_c}\right)^{\frac{1}{2}}, \tag{1.2}$$

where $\gamma^2 < 1/\sqrt{3}$. If one were only to allow two-dimensional disturbances $(k_c + M, 0)$, then the range of stable solutions would be given by Eckhaus (1965):

$$L^2 < \frac{1}{3}\left(\frac{3\pi^2}{8}\frac{Ra - Ra_c}{Ra_c}\right)^{\frac{1}{2}}. \tag{1.3}$$

Therefore, it would seem that the stability criterion is not enough to determine the state of motion; it is sufficient to determine that the motion consists of a single roll (see Schlüter, Lortz & Busse 1965) but insufficient to establish further selection among the given range. Yet in experiments, if one allows the solution to grow from the infinitesimal perturbations natural to the fluid, the state reached is a steady cell-like structure with a scale corresponding to the most critical mode.

Newell & Whitehead (1969) suggested that it would be difficult for a sideband mode solution to remain stable while still in a state of growth. Consider the following conceptual experiment: we begin the motion with a single sideband mode which if left alone will grow to a steady state; however, if at some stage in its evolution we switch on all the allowable perturbations in the fluid, then the range of stability of the sideband mode (the range of L) decreases to zero as the mode L solution is perturbed in earlier stages of its evolution. This is verified by experiment (Chen & Whitehead 1968), who found that in order to attain a solution corresponding to a sideband mode the motion must be forced externally (by use of a grid corresponding to the desired sideband) *before the Rayleigh number is allowed to cross to a supercritical value.*

The analysis to date has used a discrete multimodal description. It has dealt solely with questions of stability and not with the initial value problem in which all solutions are allowed to compete from some initial time on an equal basis. Segel (1966) did attempt to answer the question from this viewpoint, but did not allow for the relevant non-linear interactions. We wish to pose the problem in a somewhat different way, which we believe more closely describes the physical situation. Instead of beginning with a discrete number of modes, the number of which rapidly inflates due to non-linear coupling, we pose the problem as a statistical initial value problem. Given a supercritical Rayleigh number at some time $t = 0$ and an initial small random disturbance field, we seek to describe the time evolution of the statistical moments of the process. Instead of using the Boussinesq equations, we use a derivative of these equations obtained recently by Newell & Whitehead (1969).† In this derivation the basic idea is to treat the amplitude of the neutral solution given by the linear stability problem as a slowly varying function of both position and time. In that way we generate a non-linear partial differential equation for this amplitude. This equation has as special solutions the unimodal solutions discussed above, but also describes the time evolution of a spatially dependent initial profile whose Fourier synthesis would find energy continuously distributed among all the sideband modes. We make a further simplification to our model by restricting ourselves to the case of two dimensions.

To be precise, we write our neutral solution for the zeroth-order vertical velocity component $\omega(x, z, t)$ of the flow field as

$$\omega_0(x,z,t) = \left(\frac{Ra - Ra_c}{Ra_c}\right)^{\frac{1}{2}} \{W(X,T)\, e^{ik_i x} + (*)\} \sin \pi z, \tag{1.4}$$

where
$$X \propto \left(\frac{Ra - Ra_c}{Ra_c}\right)^{\frac{1}{2}} x, \quad T \propto \frac{Ra - Ra_c}{Ra_c} t;$$

$*$ refers to the complex conjugate. In order to solve the Boussinesq equations successively so that ω_0 represents the first term of a uniformly (in space and time) valid asymptotic expansion for ω in powers of $((Ra - Ra_c)/Ra_c)^{\frac{1}{2}}$, we find that

† This derivation has also been obtained by Segel (1969).

a certain solvability criterion must be met. In non-dimensional and normalized form this criterion is

$$\frac{\partial W}{\partial T} - \frac{\partial^2 W}{\partial X^2} = W - W^2 W^*. \tag{1.5}$$

For further details we refer the reader to the Newell & Whitehead paper.

From (1.5) we form the hierarchy of equations for the statistical moments (equivalently the cumulants) whose evolution in time we wish to examine. We assume the field to be spatially homogeneous in X, which means that the moments depend only on the relative and not the absolute position of the spatial arguments. As is well known in non-linear stochastic processes, the time rate of change of a cumulant of a given order depends on cumulants of a higher order, and one is faced with the usual closure difficulty fundamental to non-linear random processes. Since there is no *a priori* reason why the statistical distribution should not require all of its moments to describe its evolution from a given initial state, one is left with an infinite set of equations to solve. The closure we succeed in obtaining depends on the smallness of the initial disturbance field. If we begin with the cumulants small we find that initially they grow exponentially in time; however, the crucial point is that even though all the cumulants grow exponentially there is a certain ordering in the initial rate of growth of the different cumulants. The rate of growth of the second-order cumulant is less than that of the square of the mean; the rate of growth of the third-order cumulant is less than the growth of the product of the mean and second-order cumulants, which in turn is less than the cube of the mean. Even though the inner (initial) expansion becomes non-uniform after a certain time, the size to which the cumulants have grown remains inversely proportional to their order. Thus, using the concept of matched asymptotic expansions, the hierarchy of equations for long time turn out to be non-linear but closed. Fortunately it transpires that the outer solution is uniformly valid for all time. If ϵ ($|\epsilon| \ll 1$) is a measure of the amplitudes of the initial field then the relevant ordering parameter is $\beta(\epsilon) = [\log 1/|\epsilon|]^{-\frac{1}{2}}$. What happens is that the mean value grows from its initial amplitude of order ϵ to a finite steady value of unity. The second-order cumulant grows from its initial order ϵ^2 to a size β after a time $1/\beta^2$ and then decays with time to zero. Likewise the nth order cumulant grows to a size β^{n-1} at time $1/\beta^2$, and then decays to zero.

The net result is that we are left with a field which is one of perfect order. The motion is no longer random but is made up of discrete rolls whose size corresponds to that of the most critical wavelength. Initially the spectrum contained energy in the total bandwidth of interest, but the sideband modes were not able to compete effectively with the mean in deriving the potential energy from the mean temperature profile. Essentially the reason is that the interaction of a given sideband mode with the mean is weaker than its interaction with the other sideband modes. This becomes clear in the closure equations of §§2 and 3.

It is worth commenting on the necessity of using a matched asymptotic expansions approach in favour of a multiple time scale approach; the latter proved to be successful in obtaining a closure in interacting random, dispersive wave fields (Benney & Saffman 1966; Benney & Newell 1969). The essential point to stress is that in the present problem, unlike the case of random waves, the non-linear terms are not always weak; in fact after a certain time they are as important as the linear terms. The reason the multiple time scale approach is inadequate lies in the fact that the dependent functions themselves undergo order of magnitude changes in time. An analogous situation exists with the slow flow around spheres and cylinders. Because the amplitude of the velocity changes from zero in the Stokes solution to the free stream velocity in the Oseen solution the method of inner and outer expansions is necessary. The multiple scale approach is only effective in dealing with situations where the *fundamental* solution has the correct order of magnitude. One cannot begin with a Stokes solution whose constants are slowly varying functions of position, and hope to produce the Oseen solution, as we know that in the far field the momentum advection term belongs in the zeroth-order balance.

Of further interest is the fact that the solution to the statistical initial value problem transports the most heat through the layer. It has been suggested by Malkus (1954) that there exists some statistical stability criterion which selects among the class of solutions the solution which extremises some macroscopic quantity. Malkus further suggested a stronger hypothesis in order to obtain a simpler problem: namely, that the flow field, which ultimately occurs, not only has the maximum heat transport of all solutions of the Boussinesq equations, but may be close to the maximum heat transport of all flow fields restricted only to satisfy the boundary conditions, continuity and certain power integrals of the Boussinesq equations. Whether or not the Malkus hypothesis is true, the idea stimulated Howard (1963) to ask the following formal question. If we take a set of flow fields constrained only by boundary conditions, continuity, and certain power integrals, what is the maximum heat transport and corresponding solution over this set? The result is certainly an upper bound of the heat transport actually realized by the fluid and as such is useful when there is no other way to acquire information for turbulent convection flow. (Malkus & Howard were interested in large Rayleigh numbers. In the past year Busse 1969 has extended much of Howard's work to other turbulent flow situations.) The difficulty with the Malkus–Howard approach is that one never knows how close the actual solution comes to the upper bound. It is of some interest, therefore, that we show (§5) how the final steady solution obtained from the statistical initial value problem is the very same solution as the flow field that upper-bounds the heat transport, and is constrained not by (1.5), but only by the simplest power integral derived from (1.5).

In order to show consistancy with the discrete multimodal stability analysis, we verify in §4 that, if we take as initial conditions the finite amplitude discrete sideband mode plus some noise (instead of having only noise), then the criterion that the disorder decays is exactly the Eckhaus stability criterion (1.3). (The Schlüter *et al.* instabilities are of a three-dimensional character and so do not appear.) Thus the appearance of the single roll solution corresponding to the most critical mode depends in some sense on the relative disorder in the initial conditions. If there is such disorder that all parts of the finite bandwidth spectrum compete in some sense on an equal basis, then there seems to be a natural statistical selection process which forgets about the initial data. If, on the other hand, the initial spectrum is weighted sufficiently in favour of a discrete side-band mode, then the solution depends solely on these initial conditions.

2. Perturbation procedures

As the reader is no doubt well aware, the study of stochastic processes often involves a great deal of notation and algebraic manipulation. Unfortunately, this situation becomes even more complicated when a given problem is attacked by perturbation methods. For this reason we wish to devote this section to a detailed discussion of a somewhat simplified version of the problem posed in (1.5) equivalent to taking W real. This will permit us to make clear the critical points in the analysis while minimizing the notational difficulties and algebraic manipulations. Subsequent sections will be concerned with the more general problem.

To be specific we shall examine solutions $u(x,t)$ of the following quasilinear parabolic differential equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u - u^3, \tag{2.1}$$

involving only one space variable x, where $-\infty < x < \infty$, and the time t. As our interest lies in the solution of the statistical initial value problem we assume that $u(x,t)$ is a real-valued stationary random function of x, whose initial mean value properties are known. The stipulation of spatial homogeneity is a common practice in statistical theories of turbulence. Our effort, then, will be directed toward determining the time evolution of the mean value properties of $u(x,t)$.

The particular mean value properties that we wish to study are the physical space correlations of $u(x,t)$. As u is a stationary random function of position, these correlations depend only on the relative geometry. For example, the first-order correlation or mean is defined by

$$\langle u(x,t)\rangle = R^{(1)}(t), \tag{2.2}$$

where the angle brackets denote an ensemble average. The nth order correlations ($n \geqslant 2$) are defined by the ensemble averages,

$$\langle u(x,t)\, u(x+r,t)\, u(x+r',t) \ldots u(x+r^{(n-2)},t)\rangle, \tag{2.3}$$

which are functions of the spatial separations $r, r', \ldots, r^{(n-2)}$ and the time t. Allowing a non-zero mean implies that the correlations defined in equation (2.3) will have generalized functions as Fourier transforms. As is well known, this difficulty is avoided by introducing the nth order physical space cumulants. These cumulants have the property that they will tend to zero as the relative separations tend to infinity, and hence the cumulant transforms will be ordinary functions. Thus, for $n = 2$,

$$R^{(2)}(r,t) = \langle u(x,t)\, u(x+r,t)\rangle - \langle u(x)\rangle^2, \tag{2.4}$$

for $n = 3$,

$$R^{(3)}(r,r',t) = \langle u(x,t)\, u(x+r,t)\, u(x+r',t)\rangle - \langle u(x,t)\, u(x+r,t)\rangle \langle u(x,t)\rangle$$
$$- \langle u(x,t)\, u(x+r',t)\rangle \langle u(x,t)\rangle - \langle u(x+r,t)\, u(x+r',t)\rangle \langle u(x,t)\rangle + 2\langle u(x,t)\rangle^3, \tag{2.5}$$

and, in general,

$$R^{(n)}(r, r', \ldots, r^{(n-2)}, t) = \langle u(x,t)\, u(x+r,t)\, u(x+r',t) \ldots u(x+r^{(n-2)},t)\rangle$$
$$- \sum_{\alpha\beta\gamma\ldots} \langle u(x,t) \ldots u(x+r^{(\alpha)},t)\rangle \langle u(x+r^{(\beta)},t) \ldots u(x+r^{(\gamma)},t)\rangle \ldots, \quad (2.6)$$

where the summation contains all the necessary combinations of products of correlations, involving $u(x,t), u(x+r,t), \ldots, u(x+r^{(n-2)},t)$, to ensure the proper behaviour of $R^{(n)}$ as the relative separations tend to infinity.

The nth order Fourier space cumulants are defined by the equation,

$$R^{(n)}(r, r', \ldots, r^{(n-2)}, t) = \int_{-\infty}^{\infty} Q^{(n)}(k_l, k_{l'}, \ldots, k_{l^{(n-2)}}, t)$$
$$\times \exp(ik_l r + ik_{l'} r' + \ldots + ik_{l^{(n-2)}} r^{(n-2)})\, dk_l\, dk_{l'} \ldots dk_{l^{(n-2)}}. \quad (2.7)$$

The time evolution of the physical space cumulants is governed by an infinite set of coupled equations obtained by properly averaging (2.1). We shall derive the first two equations of this set and then indicate the form of the general expression. The first equation is obtained by directly averaging (2.1), which can be written as

$$\left\langle \frac{\partial u(x,t)}{\partial t} \right\rangle - \left\langle \frac{\partial^2 u(x,t)}{\partial x^2} \right\rangle = \langle u(x,t)\rangle - \langle u(x,t)\, u(x,t)\, u(x,t)\rangle. \quad (2.8)$$

Utilizing the fact that the operations of averaging and differentiating may readily be shown to commute (see Batchelor 1953), (2.8) becomes

$$\frac{\partial \langle u(x,t)\rangle}{\partial t} - \frac{\partial^2 \langle u(x,t)\rangle}{\partial x^2} = \langle u(x,t)\rangle - \langle u(x,t)\, u(x,t)\, u(x,t)\rangle. \quad (2.9)$$

Now we must introduce the expressions for the physical space cumulants given in (2.2)–(2.6). Because of the spatial homogeneity assumption, the second term on the left of (2.9) is zero, and we thus obtain

$$\frac{\partial R^{(1)}(t)}{\partial t} - R^{(1)}(t) = -R^{(3)}(0,0,t) - 3R^{(2)}(0,t)\, R^{(1)}(t) - \overset{3}{R^{(1)}}(t). \quad (2.10)$$

Notice that this equation, which we shall interpret as the governing equation for the time evolution of $R^{(1)}$, contains the second- and third-order physical space cumulants.

The second member of our set is obtained by first multiplying (2.1) by $u(x',t)$ and averaging. Thus, we have

$$\left\langle u(x',t) \frac{\partial u(x,t)}{\partial t} \right\rangle - \left\langle u(x',t) \frac{\partial^2 u(x,t)}{\partial x^2} \right\rangle = \langle u(x',t)\, u(x,t)\rangle - \langle u(x',t)\, u^3(x,t)\rangle, \quad (2.11)$$

where x and x' are unrelated. Next we must express (2.1) in terms of x', multiply by $u(x,t)$ and average to obtain

$$\left\langle u(x,t) \frac{\partial u(x',t)}{\partial t} \right\rangle - \left\langle u(x,t) \frac{\partial^2 u(x',t)}{\partial x'^2} \right\rangle = \langle u(x,t)\, u(x',t)\rangle - \langle u(x,t)\, u^3(x',t)\rangle. \quad (2.12)$$

Upon setting $x' = x+r$ and adding (2.11) and (2.12) we have

$$\frac{\partial\langle u(x,t)\,u(x+r,t)\rangle}{\partial t} - \left\langle u(x+r,t)\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\frac{\partial^2 u(x+r)}{\partial r^2}\right\rangle - 2\langle u(x,t)\,u(x+r,t)\rangle$$
$$= -\langle u(x+r,t)\,u(x,t)\,u(x,t)\,u(x,t)\rangle - \langle u(x,t)\,u(x+r,t)\,u(x+r,t)\,u(x+r,t)\rangle. \quad (2.13)$$

Combining the expression,

$$\frac{\partial R^{(2)}(r,t)}{\partial t} = \frac{\partial}{\partial t}\langle u(x,t)\,u(x+r,t)\rangle - 2R^{(1)}(t)\frac{\partial R^{(1)}(t)}{\partial t} \quad (2.14)$$

with (2.10) and (2.13) leads to the second member of our set: namely,

$$\frac{\partial R^{(2)}(r,t)}{\partial t} - 2\frac{\partial^2 R^{(2)}(r,t)}{\partial r^2} - 2R^{(2)}(r,t) = -\mathscr{P}_{\underset{2}{r-r}}\{R^{(4)}(0,0,r,t) + 3R^{(3)}(0,r,t)\,R^{(1)}(t)$$
$$+3R^{(2)}(r,t)\,R^{(2)}(0,t) + 3R^{(2)}(r,t)\,R^{(1)}(t)\}, \quad (2.15)$$

where the symbol $\mathscr{P}_{rr'\ldots r^{(m)}}$ is used to imply a cyclic summation over $r, r', \ldots, r^{(m)}$. Thus, the equation governing the time evolution of $R^{(2)}$ contains $R^{(3)}$ and $R^{(4)}$. By similar manipulations we can obtain the governing equation for the general nth order physical space cumulant. It will have the form (for $n \geqslant 2$),

$$\left[\frac{\partial}{\partial t} - 2\sum_{j=0}^{n-2}\frac{\partial^2}{\partial r^{(j)2}} - 2\sum_{0\leqslant j<k\leqslant n-2}\frac{\partial^2}{\partial r^{(j)}\,\partial r^{(k)}} - n\right] R^{(n)}(r, r', \ldots, r^{(n-2)}, t)$$
$$= -\mathscr{P}_{\underset{2}{r\ldots}}\{R^{(n+2)}(0,0,r,\ldots,r^{(n-2)},t) + 3R^{(n+1)}(0,r,r',\ldots,r^{(n-2)},t)\,R^{(1)}(t)$$
$$+3R^{(n)}(r,r',\ldots,r^{(n-2)},t)\,R^{(1)}(t) + 3R^{(n)}(r,r',\ldots,r^{(n-2)},t)\,R^{(2)}(0,t)$$
$$+3R^{(n)}(0,r',\ldots,r^{(n-2)},t)\,R^{(2)}(r,t) + \ldots + 3R^{(n)}(r,r',\ldots,r^{(n-3)},0,t)\,R^{(2)}(r^{(n-2)},t)$$
$$+\text{products of cumulants of order less than } n\}. \quad (2.16)$$

A few comments are in order regarding (2.16). The left-hand side is a linear parabolic operator in $n-1$ space dimensions. The term $nR^{(n)}$ arises from our having combined n equations of the form,

$$\left\langle u(x+r,t)\,u(x+r',t)\ldots u(x+r^{(n-2)},t)\frac{\partial u(x,t)}{\partial t}\right\rangle$$
$$-\left\langle u(x+r,t)\,u(x+r',t)\ldots u(x+r^{(n-2)},t)\frac{\partial^2 u(x,t)}{\partial x^2}\right\rangle$$
$$= \langle u(x,t)\,u(x+r,t)\ldots u(x+r^{(n-2)},t)\rangle - \langle u(x,t)\,u(x,t)\,u(x,t)$$
$$\times\, u(x+r,t)\ldots u(x+r^{(n-2)},t)\rangle,$$

in order to construct an equation for $\partial R^{(n)}/\partial t$ as was done above for $\partial R^{(2)}/\partial t$. Finally, we note that the right-hand side of (2.16) involves $R^{(n+1)}$ and $R^{(n+2)}$. It is this latter feature which makes the solution of the statistical initial value problem so difficult. An infinite system of equations must be considered for a simultaneous determination of all cumulants. (In order to close off, or circumvent, the infinite set of cumulant equations one would, in general, be forced to resort to systematic, although mathematically unjustifiable, analytical approximations.)

Fortunately, in the special case when the cumulants are initially small, we are able to obtain an exact solution of the cumulant equations by perturbation methods. To be specific we shall consider the statistical initial value problem posed by (2.10) and (2.16) with the following initial conditions:

$$R^{(1)}(0) = \epsilon, \quad R^{(n)}(r, r', \ldots, r^{(n-2)}, 0) = \epsilon^n h^{(n)}(r, r', \ldots, r^{(n-2)}) \quad (n \geqslant 2), \tag{2.17}$$

where ϵ is a real constant, $0 < |\epsilon| \ll 1$. The functions $h^{(n)}$ are assumed to be independent of ϵ, and their Fourier transforms are defined by

$$h^{(n)}(r, r', \ldots, r^{(n-2)}) = \int_{-\infty}^{\infty} H^{(n)}(k_l, k_{l'}, \ldots, k_{l^{(n-2)}}) \times \exp(ik_l r + ik_{l'} r' + \ldots + ik_{l^{(n-2)}} r^{(n-2)})\, dk_l \ldots dk_{l^{(n-2)}}. \tag{2.18}$$

Our first task is the construction of what we shall term the inner asymptotic expansions for the cumulants. We expect that these expansions will be useful only for a limited time. However, as is common in singular perturbation problems, the behaviour of these inner expansions will suggest the appropriate manner of rescaling the differential equations so that the solutions can be extended to later times. The leading term in the inner expansion for $R^{(n)}$, $n \geqslant 1$, will be determined by setting the right-hand side of (2.10) and (2.16) equal to zero. The higher order terms, in ϵ, will then be obtained by successively iterating in (2.10) and (2.16) upon the lower order terms. It is clear from (2.10) and (2.16) and the initial conditions in (2.17) that the inner expansions will have the form,

$$R^{(n)}(r, r', \ldots, r^{(n-2)}, t) \sim \epsilon^n \sum_{j=0}^{\infty} \epsilon^{2j} R_j^{(n)}(r, r', \ldots, r^{(n-2)}, t) \quad (n \geqslant 1), \tag{2.19}$$

where $R_j^{(n)}$ is independent of ϵ. Consistent with (2.17) we shall require that

$$R_0^{(1)}(0) = 1, \quad R_j^{(1)}(0) = 0 \quad \text{for} \quad j \geqslant 1, \quad R_0^{(n)}(r, r', \ldots, r^{(n-2)}, 0) = h^{(n)}(r, r', \ldots, r^{(n-2)}), \quad R_j^{(n)}(r, r', \ldots, r^{(n-2)}, 0) = 0 \quad \text{for} \quad (n \geqslant 2, j \geqslant 1). \tag{2.20}$$

In order better to analyze the behaviour of the inner expansions, it proves convenient to Fourier transform the equations for the physical space cumulants using (2.7). For notational convenience we shall define

$$Q^{(1)}(t) = R^{(1)}(t). \tag{2.21}$$

The equations for the Fourier space cumulants thus can be written:

$$\frac{\partial Q^{(1)}}{\partial t} - Q^{(1)} = -\int Q^{(3)}_{mp}\, dk_{mp} - 3Q^{(1)} \int Q^{(2)}_m\, dk_m - Q^{(1)^3}, \tag{2.22}$$

$$\frac{\partial Q^{(2)}_l}{\partial t} + 2\sigma_l Q^{(2)}_l - 2Q^{(2)}_l = -\mathscr{P}_{ll'} \left\{ \int Q^{(4)}_{mp(l-m-p)}\, dk_{mp} + 3Q^{(1)} \int Q^{(3)}_{m(l-m)}\, dk_m + 3Q^{(2)}_l \int Q^{(2)}_m\, dk_m + 3Q^{(2)}_l Q^{(1)^2} \right\} \quad (k_l + k_{l'} = 0), \tag{2.23}$$

$$
\begin{aligned}
\frac{\partial Q^{(3)}_{ll'}}{\partial t} + \sigma_{ll'l''} Q^{(3)}_{ll'} - 3Q^{(3)}_{ll'} = -\mathscr{P}_{ll'l''}\Big\{ & \int Q^{(5)}_{l'mp(l-m-p)}\, dk_{mp} + 3Q^{(1)} \int Q^{(4)}_{l'm(l-m)}\, dk_m \\
& + 3Q^{(3)}_{ll'} \int Q^{(2)}_{m}\, dk_m + 3Q^{(2)}_{l'} \int Q^{(3)}_{l''m}\, dk_m + 3Q^{(2)}_{l''} \int Q^{(3)}_{ml'}\, dk_m \\
& + 3Q^{(3)}_{ll'} {Q^{(1)}}^2 + 6Q^{(1)} Q^{(2)}_{l'} Q^{(2)}_{l''} \Big\} \quad (k_l + k_{l'} + k_{l''} = 0),
\end{aligned} \tag{2.24}
$$

$\vdots$

$$
\begin{aligned}
& \frac{\partial Q^{(n)}_{ll'\ldots l(n-2)}}{\partial t} + \sigma_{ll'\ldots l(n-1)} Q^{(n)}_{ll'\ldots l(n-2)} - nQ^{(n)}_{ll'\ldots l(n-2)} \\
& \quad = -\mathscr{P}_{ll'\ldots l(n-1)} \Big\{ \int Q^{(n+2)}_{l'l''\ldots l(n-2)\, mp(l-m-p)}\, dk_{mp} + 3Q^{(1)} \int Q^{(n+1)}_{l'l''\ldots l(n-2)\, m(l-m)}\, dk_m \\
& \qquad + 3Q^{(n)}_{ll'\ldots l(n-2)} {Q^{(1)}}^2 + 3Q^{(n)}_{ll'\ldots l(n-2)} \int Q^{(2)}_{m}\, dk_m + 3Q^{(2)}_{l'} \int Q^{(n)}_{ml''l'''\ldots l(n-1)}\, dk_m \\
& \qquad + \ldots + 3Q^{(2)}_{l(n-1)} \int Q^{(n)}_{l'l''\ldots l(n-2)m}\, dk_m \\
& \qquad + \text{products of cumulants of order less than } n \Big\} \\
& \qquad\qquad (k_l + k_{l'} + \ldots + k_{l(n-1)} = 0),
\end{aligned} \tag{2.25}
$$

where the following condensed notation has been adopted:

$$
\left.
\begin{aligned}
& Q^{(n)}_{ll'\ldots l(n-2)} = Q^{(n)}(k_l, k_{l'}, \ldots, k_{l(n-2)}, t), \quad dk_{m_1 m_2 \ldots m_r} = \Pi^r_{p=1}\, dk_{m_p}, \\
& \sigma_{l_1 l_2 \ldots l_r, m_1 m_2 \ldots m_s} = \sum_{\alpha=1}^{r} k^2_{l_\alpha} - \sum_{\beta=1}^{s} k^2_{m_\beta},
\end{aligned}
\right\} \tag{2.26}
$$

and the implied limits on all integrations are from $-\infty$ to $+\infty$. Further $\mathscr{P}_{ll'\ldots l(m)}$ implies a cyclic summation over the wave numbers $k_l, k_{l'}, \ldots, k_{l(m)}$.

Corresponding to the asymptotic expansions for the physical space cumulants given in (2.19), the Fourier space cumulants will have an inner expansion

$$
Q^{(n)}_{ll'\ldots l(n-2)} \sim \epsilon^n \sum_{j=0}^{\infty} \epsilon^{2j} Q^{(n)}_{jll'\ldots l(n-2)} \tag{2.27}
$$

where the j subscript denotes the perturbation ordering. The leading term in each expansion is obtained by equating the right-hand side of (2.22)–(2.25) to zero. Making use of the initial conditions given by (2.17) and (2.18) we easily find that

$$
Q^{(1)}_{0} = e^t, \tag{2.28}
$$

$$
\left.
\begin{aligned}
Q^{(n)}_{0ll'\ldots l(n-2)} &= H^{(n)}_{ll'\ldots l(n-2)} \exp\,(nt - \sigma_{ll'\ldots l(n-1)} t), \\
n \geqslant 2, \quad & k_l + k_{l'} + \ldots + k_{l(n-1)} = 0.
\end{aligned}
\right\} \tag{2.29}
$$

A steepest descent analysis on the corresponding cumulants in physical space shows that the fastest growth rate occurs in the $k_l = k_{l'} = \ldots = k_{l(n-2)} = 0$ mode. The higher order perturbation terms are found by successively iterating in (2.22)–(2.25) upon the lower order terms. For our purposes it is sufficient to calculate only the expressions for $Q^{(n)}_1$, $n \geqslant 1$. This is accomplished by substituting $Q^{(n)}_1$ into the left-hand side of (2.22)–(2.25) and by replacing each $Q^{(p)}$ by the previously computed $Q_0^{(p)}$ in the right-hand side of these same equations. Solving this set of equations subject to the condition that $Q_1^{(n)} = 0$ when $t = 0$, $n \geqslant 1$, we obtain

$$Q_1^{(1)} = -e^t \left\{ \int H^{(3)}_{mp} \frac{\exp(2t - \sigma_{mp(m+p)}t) - 1}{2 - \sigma_{mp(m+p)}} dk_{mp} + 3 \int H^{(2)}_m \frac{\exp(2t - 2\sigma_m t) - 1}{2 - 2\sigma_m} dk_m + \frac{e^{2t} - 1}{2} \right\}, \quad (2.30)$$

$$\begin{aligned} Q_{1l}^{(2)} = -\exp(2t - 2\sigma_l t)\, \mathscr{P}_{ll'} \Bigg\{ & \int H^{(4)}_{mp(l-m-p)} \frac{\exp(2t + \sigma_{l,mp(l-m-p)}t) - 1}{2 + \sigma_{l,mp(l-m-p)}} dk_{mp} \\ & + 3 \int H^{(3)}_{m(l-m)} \frac{\exp(2t + \sigma_{l,m(l-m)}t) - 1}{2 + \sigma_{l,m(l-m)}} dk_m \\ & + 3H_l^{(2)} \int H^{(2)}_m \frac{\exp(2t - 2\sigma_m t) - 1}{2 - 2\sigma_m} dk_m + 3H_l^{(2)} \frac{e^{2t} - 1}{2} \Bigg\}, \end{aligned}$$

$$k_l + k_{l'} = 0, \quad (2.31)$$

$\vdots$

$$\begin{aligned} Q^{(n)}_{1ll'\ldots l^{(n-2)}} = -\exp(nt - \sigma_{ll'\ldots l^{(n-1)}}t)\, \mathscr{P}_{ll'\ldots l^{(n-1)}} \Bigg\{ & \int H^{(n+2)}_{l'l''\ldots l^{(n-1)}mp(l-m-p)} \\ \times \frac{\exp(2t + \sigma_{l,mp(l-m-p)}t) - 1}{2 + \sigma_{l,mp(l-m-p)}} dk_{mp} & + 3 \int H^{(n+1)}_{l'l''\ldots l^{(n-2)}m(l-m)} \frac{\exp(2t + \sigma_{l,m(l-m)}t) - 1}{2 + \sigma_{l,m(l-m)}} dk_m \\ & + 3H^{(n)}_{ll'\ldots l^{(n-2)}} \frac{e^{2t} - 1}{2} + 3H^{(n)}_{ll'\ldots l^{(n-2)}} \int H^{(2)}_m \frac{\exp(2t - 2\sigma_m t) - 1}{2 - 2\sigma_m} dk_m \\ & + 3H^{(2)}_{l'} \int H^{(n)}_{ml''l'''\ldots l^{(n-2)}} \frac{\exp(2t + \sigma_{l,l'm(l+l'-m)}t) - 1}{2 + \sigma_{l,l'm(l+l'-m)}} dk_m + \ldots \\ & + 3H^{(2)}_{l^{(n-1)}} \int H^{(n)}_{l'l''\ldots l^{(n-2)}m} \frac{\exp(2t + \sigma_{l,l^{(n-1)}m(l+l^{(n-1)}-m)}t) - 1}{2 + \sigma_{l,l^{(n-1)}m(l+l^{(n-1)}-m)}} dk_m \\ & + \text{products of cumulants of order less than } n \Bigg\}, \end{aligned}$$

$$k_l + k_{l'} + \ldots + k_{l^{(n-1)}} = 0. \quad (2.32)$$

Having determined the first two terms in the inner expansion for each Fourier space cumulant, we now can transform back to physical space. With the aid of (2.28)–(2.29) we have that

$$\left. \begin{aligned} & R_0^{(1)}(t) = e^t, \\ & R_0^{(n)}(r, r', \ldots, r^{(n-2)}, t) = \exp(nt) \int H^{(n)}_{ll'\ldots l^{(n-2)}} \\ & \qquad \times \exp(-\sigma_{ll'\ldots l^{(n-1)}}t) \exp(ik_l r + ik_{l'} r' + \ldots + ik_{l^{(n-2)}} r^{(n-2)}) \\ & \qquad \times \delta_{ll'\ldots l^{(n-1)}}\, dk_{ll'\ldots l^{(n-1)}} \quad (n \geqslant 2), \end{aligned} \right\} \quad (2.33)$$

where $\delta_{ll'\ldots l^{(n-1)}}$ is a Dirac delta function with argument $k_l + k_{l'} + \ldots + k_{l^{(n-1)}}$.

By a simple steepest descent analysis we can show that the behaviour of $R_0^{(n)}$ as $t \to +\infty$ is given by (assuming that each $H^{(n)}$ is non-zero at the origin)

$$\left.\begin{aligned} R_0^{(n)}(r, r', \ldots, r^{(n-2)}, t) &\sim \frac{1}{\sqrt{n}}\left(\frac{\pi}{t}\right)^{(n-1)/2} H^{(n)}(0,0,\ldots,0)\, e^{nt} \\ &\times \exp\left\{\left[(1-n)\sum_{j=0}^{n-2} r^{(j)^2} + 2\sum_{0\leqslant j<k\leqslant n-2} r^{(j)} r^{(k)}\right]\Big/ 4nt\right\} \quad (n \geqslant 2), \\ \text{for} \quad &\frac{r}{t^{\frac{1}{2}}}, \frac{r'}{t^{\frac{1}{2}}}, \ldots, \frac{r^{(n-2)}}{t^{\frac{1}{2}}} \leqslant O(1). \end{aligned}\right\} \tag{2.34}$$

The $1/t^{\frac{1}{2}(n-2)}$ decay rate is characteristic of diffusion problems. As a result of it we have, for example, that

$$\frac{R_0^{(m)} R_0^{(p)}}{R_0^{(n)}} = O(t^{\frac{1}{2}}) \quad \text{for} \quad t \to +\infty \quad \text{with} \quad m+p = n. \tag{2.35}$$

This result will be of utmost importance in the ensuing analysis.

Next we shall determine the long time behaviour of the $R_1^{(n)}$ terms. First, let us examine $R_1^{(1)}$ as given by (2.30). Again applying a simple steepest descent analysis it can be shown that the term involving $H^{(3)}$ is $O(e^{3t}/t)$ and the one involving $H^{(2)}$ is $O(e^{3t}/t^{\frac{1}{2}})$. The last term is clearly $O(e^{3t})$. In order of magnitude form then, the inner expansion behaves like

$$R^{(1)}(t) = \epsilon e^t\{1 + O(\epsilon^2 e^{2t}) + O(\epsilon^2 e^{2t}/t^{\frac{1}{2}}) + O(\epsilon^2 e^{2t}/t) + \ldots\} \tag{2.36}$$

for $t \gg 1$. Thus, the inner asymptotic expansion becomes non-uniform when

$$\epsilon^2 e^{2t} = O(1) \quad \text{or} \quad t = O(1/\beta^2) \quad \text{with} \quad \beta \equiv 1/(\log 1/|\epsilon|)^{\frac{1}{2}}. \tag{2.37}$$

For values of t in this range, the inner expansion suggests that

$$R^{(1)}(t) = O(1). \tag{2.38}$$

Transforming the expression for $Q_1^{(n)}$, given in (2.32), back into physical space in order to determine the long time behaviour of $R_1^{(n)}$ requires slightly more complicated steepest descent calculations. However, it turns out that the term involving $H^{(n)}e^{2t}$ and some of the other terms involving only cumulants of order less than n lead to the largest contributions. In order of magnitude form, the general inner expansion can be written as

$$R^{(n)}(r, r', \ldots, r^{(n-2)}, t) = \frac{\epsilon^n e^{nt}}{t^{(n-1)/2}}\left\{O(1) + O(\epsilon^2 e^{2t}) + O\left(\epsilon^2 \frac{e^{2t}}{\sqrt{t}}\right) + \right\} \quad \text{for} \quad t \gg 1 \quad (n \geqslant 1). \tag{2.39}$$

Thus, the inner expansion for each physical space cumulant becomes disordered on the same time scale; namely,

$$t = O(1/\beta^2).$$

Further the inner expansion indicates that $R^{(n)} = O(\beta^{n-1})$ for $t = 1/\beta^2$. It should be recalled that when $t = 0$, $R^{(n)} = O(\epsilon^n)$. Consequently, by the time the inner expansions become disordered, the cumulants have become, in an asymptotic sense, transcendentally large compared to their initial values.

In order to ascertain the behaviour of the cumulants for $t \geqslant O(1/\beta^2)$ we shall employ a matched asymptotic expansions approach; this technique for constructing a uniformly valid solution in stochastic problems was first used by Benney & Lange (1969). We rescale the cumulants based on the orders of magnitude suggested by the inner expansions; namely, $R^{(n)} = O(\beta^{n-1})$ for $t = O(1/\beta^2)$. We define

$$R^{(n)}(r, r', \ldots, r^{(n-2)}, t) = \beta^{n-1} \tilde{R}^{(n)}(r, r', \ldots, r^{(n-2)}, t) \quad (n \geqslant 1), \tag{2.40}$$

and the corresponding Fourier space cumulant,

$$Q^{(n)}_{ll'\ldots l^{(n-2)}} = \beta^{n-1} \tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}} \quad (n \geqslant 1). \tag{2.41}$$

Substituting this expression into (2.22)–(2.25), the rescaled equations become

$$\frac{\partial \tilde{Q}^{(1)}}{\partial t} - \tilde{Q}^{(1)} + \overset{3}{\tilde{Q}^{(1)}} = -3\beta \tilde{Q}^{(1)} \int \tilde{Q}^{(2)}_m \, dk_m - \beta^2 \int \tilde{Q}^{(3)}_{mp} \, dk_{mp} \tag{2.42}$$

$$\frac{\partial \tilde{Q}^{(2)}_l}{\partial t} + 2\sigma_l \tilde{Q}^{(2)}_l - 2\tilde{Q}^{(2)}_l + 6\tilde{Q}^{(2)}_l \overset{2}{\tilde{Q}^{(1)}} = -\mathscr{P}_{ll'} \left\{ 3\beta \tilde{Q}^{(1)} \int \tilde{Q}^{(3)}_{m(l-m)} \, dk_m \right.$$
$$\left. + 3\beta \tilde{Q}^{(2)}_l \int \tilde{Q}^{(2)}_m \, dk_m + \beta^2 \int \tilde{Q}^{(4)}_{mp(l-m-p)} \, dk_{mp} \right\}, \quad (k_l + k_{l'} = 0), \tag{2.43}$$

$$\frac{\partial \tilde{Q}^{(3)}_{ll'}}{\partial t} + \sigma_{ll'l''} \tilde{Q}^{(3)}_{ll'} - 3\tilde{Q}^{(3)}_{ll'} + 9\tilde{Q}^{(3)}_{ll'} \overset{2}{\tilde{Q}^{(1)}} + 6\tilde{Q}^{(1)} \mathscr{P}_{ll'l''} \tilde{Q}^{(2)}_{l'} \tilde{Q}^{(2)}_{l''}$$
$$= -\mathscr{P}_{ll'l''} \left\{ 3\beta \tilde{Q}^{(1)} \int \tilde{Q}^{(4)}_{l'm(l-m)} \, dk_m + 3\beta \tilde{Q}^{(3)}_{ll'} \int \tilde{Q}^{(2)}_m \, dk_m + 3\beta \tilde{Q}^{(2)}_{l'} \int \tilde{Q}^{(3)}_{l''m} \, dk_m \right.$$
$$\left. + 3\beta \tilde{Q}^{(2)}_{l''} \int \tilde{Q}^{(3)}_{l'm} \, dk_m + \beta^2 \int \tilde{Q}^{(5)}_{l'mp(l-m-p)} \, dk_{mp} \right\}, \quad (k_l + k_{l'} + k_{l''} = 0), \tag{2.44}$$

$$\vdots$$

$$\frac{\partial \tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}}}{\partial t} + \sigma_{ll'\ldots l^{(n-1)}} \tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}} - n\tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}} + 3n\tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}} \overset{2}{\tilde{Q}^{(1)}}$$
$$+ [\text{certain products of cumulants of order less than } n]$$
$$= -\mathscr{P}_{ll'\ldots l^{(n-1)}} \left\{ 3\beta \tilde{Q}^{(1)} \int \tilde{Q}^{(n+1)}_{l'l''\ldots l^{(n-2)} m(l-m)} \, dk_m + 3\beta \tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}} \right.$$
$$\times \int \tilde{Q}^{(2)}_m \, dk_m + 3\beta \tilde{Q}^{(2)}_{l'} \int \tilde{Q}^{(n)}_{ml''l'''\ldots l^{(n-1)}} \, dk_m + \ldots + 3\beta \tilde{Q}^{(2)}_{l^{(n-1)}}$$
$$\times \int \tilde{Q}^{(n)}_{l'l''\ldots l^{(n-2)} m} \, dk_m + \beta [\text{certain products of cumulants of order less than } n]$$
$$\left. + \beta^2 \int \tilde{Q}^{(n+2)}_{l'l''\ldots l^{(n-2)} mp(l-m-p)} \, dk_{mp} \right\}, \quad (k_l + k_{l'} + \ldots + k_{l^{(n-1)}} = 0). \tag{2.45}$$

The manner in which β appears in this set of equations suggests that we attempt an outer expansion for each physical space and Fourier space cumulant of the form,

$$\tilde{R}^{(n)}(r, r', \ldots, r^{(n-2)}, t) \sim \sum_{j=0}^{\infty} \beta^j(\epsilon) \tilde{R}^{(n)}_j(r, r', \ldots, r^{(n-2)}, t), \tag{2.46}$$

$$\tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}} \sim \sum_{j=0}^{\infty} \beta^j(\epsilon) \tilde{Q}^{(n)}_{jll'\ldots l^{(n-2)}} \quad \text{for} \quad n \geqslant 1. \tag{2.47}$$

Substituting these expansions into (2.42)–(2.45), we find that the $O(1)$ terms must satisfy

$$\frac{\partial \tilde{Q}_0^{(1)}}{\partial t} - \tilde{Q}_0^{(1)} + \overset{3}{\tilde{Q}}{}_0^{(1)} = 0, \tag{2.48}$$

$$\frac{\partial \tilde{Q}_{0l}^{(2)}}{\partial t} + 2\sigma_l \tilde{Q}_{0l}^{(2)} - 2\tilde{Q}_{0l}^{(2)} + 6\overset{2}{\tilde{Q}}{}_0^{(1)} \tilde{Q}_{0l}^{(2)} = 0, \tag{2.49}$$

$$\frac{\partial \tilde{Q}_{0ll'}^{(3)}}{\partial t} + \sigma_{ll'l''} \tilde{Q}_{0ll'}^{(3)} - 3\tilde{Q}_{0ll'}^{(3)} + 9\overset{2}{\tilde{Q}}{}_0^{(1)} \tilde{Q}_{0ll'}^{(3)} = -6\tilde{Q}_0^{(1)} \mathscr{P}_{ll'l''} \tilde{Q}_{0l'}^{(2)} \tilde{Q}_{0l''}^{(2)} \quad (k_l + k_{l'} + k_{l''} = 0), \tag{2.50}$$

$$\vdots$$

$$\frac{\partial \tilde{Q}_{0ll'\ldots l^{(n-2)}}^{(n)}}{\partial t} + \sigma_{ll'\ldots l^{(n-1)}} \tilde{Q}_{0ll'\ldots l^{(n-2)}}^{(n)} - n\tilde{Q}_{0ll'\ldots l^{(n-2)}}^{(n)} + 3n\overset{2}{\tilde{Q}}{}_0^{(1)} \tilde{Q}_{0ll'\ldots l^{(n-2)}}^{(n)}$$
$$= \text{certain products of the } \tilde{Q}_0^{(p)} \text{ with } p < n, \quad (k_l + k_{l'} + \ldots + k_{l^{(n-1)}} = 0). \tag{2.51}$$

The first equation in this set is non-linear, the others are linear. Their form requires us to solve them successively. This we can readily do.

$$\tilde{Q}_0^{(1)} = \frac{e^t}{[\alpha_0^{(1)} + e^{2t}]^{\frac{1}{2}}}, \tag{2.52}$$

$$\tilde{Q}_{0l}^{(2)} = \frac{\alpha_{0l}^{(2)}\, e^{2t - 2\sigma_l t}}{[\alpha_0^{(1)} + e^{2t}]^3}, \tag{2.53}$$

$$\tilde{Q}_{0ll'}^{(3)} = \frac{\alpha_{0ll'}^{(3)} \exp(3t - \sigma_{ll'l''} t)}{[\alpha_0^{(1)} + e^{2t}]^{\frac{3}{2}}} \left\{ \alpha_{0ll'}^{(3)} - 6\mathscr{P}_{ll'l''} \alpha_{0l'}^{(2)} \alpha_{0l''}^{(2)} \int_{t_0}^{t} \frac{\exp(2s + \sigma_{l,l'l''} s)}{[\alpha_0^{(1)} + e^{2s}]^2} ds \right\}, \tag{2.54}$$

$$\vdots$$

$$\tilde{Q}_{0ll'\ldots l^{(n-2)}}^{(n)} = \frac{\exp(nt - \sigma_{ll'\ldots l^{(n-1)}} t)}{[\alpha_0^{(1)} + e^{2t}]^{3n/2}} \{\alpha_{0ll'\ldots l^{(n-2)}}^{(2)} + \text{terms involving certain}$$
$$\text{of the } \alpha_0^{(p)} \text{ functions with } p < n\} \quad (n \geqslant 2). \tag{2.55}$$

The functions $\alpha_0^{(n)}$ must be determined by matching to the inner expansion. Before doing this we shall determine the expressions for $\tilde{Q}_1^{(1)}$ and $\tilde{Q}_1^{(2)}$. From (2.42) and (2.47) we see that $\tilde{Q}_1^{(1)}$ must satisfy

$$\frac{\partial \tilde{Q}_1^{(1)}}{\partial t} - \tilde{Q}_1^{(1)} + 3\overset{2}{\tilde{Q}}{}_0^{(1)} \tilde{Q}_1^{(1)} = -3\tilde{Q}_0^{(1)} \int \tilde{Q}_{0m}^{(2)}\, dk_m, \tag{2.56}$$

or

$$\frac{\partial \tilde{Q}_1^{(1)}}{\partial t} - \tilde{Q}_1^{(1)} + \frac{3e^{2t}}{[\alpha_0^{(1)} + e^{2t}]} \tilde{Q}_1^{(1)} = -\frac{3e^t}{[\alpha_0^{(1)} + e^{2t}]^{\frac{1}{2}}} \int_{-\infty}^{\infty} \alpha_{0m}^{(2)} \frac{e^{2t - 2\sigma_m t}}{[\alpha_0^{(1)} + e^{2t}]^3} dk_m, \tag{2.57}$$

which gives us upon integration

$$\tilde{Q}_1^{(1)} = \frac{e^t}{[\alpha_0^{(1)} + e^{2t}]^{\frac{3}{2}}} \left\{ \alpha_1^{(1)} - 3\int_{t_0}^{t} \int_{-\infty}^{\infty} \alpha_{0m}^{(2)} \frac{e^{2s - 2\sigma_m s}}{[\alpha_0^{(1)} + e^{2s}]^2} dk_m\, ds \right\}. \tag{2.58}$$

The equation for $\tilde{Q}_1^{(2)}$ is obtained from (2.43):

$$\frac{\partial \tilde{Q}_{1l}^{(2)}}{\partial t} + 2\sigma_l \tilde{Q}_{1l}^{(2)} - 2\tilde{Q}_{1l}^{(2)} + 6\overset{2}{\tilde{Q}}{}_0^{(1)} \tilde{Q}_{1l}^{(2)}$$
$$= -12\tilde{Q}_0^{(1)} \tilde{Q}_1^{(1)} \tilde{Q}_{0l}^{(2)} - 3\mathscr{P}_{ll'} \left\{ \tilde{Q}_0^{(1)} \int \tilde{Q}_{0m(l-m)}^{(3)}\, dk_m + 3\tilde{Q}_{0l}^{(2)} \int \tilde{Q}_{0m}^{(2)}\, dk_m \right\}, \tag{2.59}$$

or

$$\frac{\partial}{\partial t}[\tilde{Q}^{(2)}_{1l}\,e^{2\sigma_l t-2t}(\alpha^{(1)}_0+e^{2t})^3] = -12\,\frac{\alpha^{(2)}_{0l}\alpha^{(1)}_1 e^{2t}}{[\alpha^{(1)}_0+e^{2t}]^2}+\frac{36\alpha^{(2)}_{0l}}{[\alpha^{(1)}_0+e^{2t}]^2}$$
$$\times\int_{t_0}^{t}\int_{-\infty}^{\infty}\alpha^{(2)}_{0m}\frac{e^{-2\sigma_m s+2s}}{[\alpha^{(1)}_0+e^{2s}]^2}\,dk_m\,ds - e^{2\sigma_l t-2t}[\alpha^{(1)}_0+e^{2t}]^3\,\mathscr{P}_{ll'}$$
$$\times\left\{\frac{3e^t}{[\alpha^{(1)}_0+e^{2t}]^{\frac{1}{2}}}\int_{-\infty}^{\infty}\left[\alpha^{(3)}_{0m(l-m)}\frac{\exp(-\sigma_{ml(l-m)}t+3t)}{[\alpha^{(1)}_0+e^{2t}]^{\frac{9}{2}}}-6\,\frac{\exp(-\sigma_{ml(l-m)}t+3t)}{[\alpha^{(1)}_0+e^{2t}]^{\frac{9}{2}}}\right.\right.$$
$$\left.\times\,\mathscr{P}_{m(l-m)l'}\,\alpha^{(2)}_{0l'}\,\alpha^{(2)}_{0(l-m)}\int_{t_0}^{t}\frac{\exp(2s+\sigma_{m,l'(l-m)}s)}{[\alpha^{(1)}_0+e^{2s}]^2}\,ds\right]dk_m$$
$$\left.+\frac{36\alpha^{(2)}_{0l}}{[\alpha^{(1)}_0+e^{2t}]^6}\int_{-\infty}^{\infty}\alpha^{(2)}_{0m}\exp(-2\sigma_{lm}t+4t)\right\}. \tag{2.60}$$

Upon integration we obtain

$$\tilde{Q}^{(2)}_{1l} = \frac{e^{2t-2\sigma_l t}}{[\alpha^{(1)}_0+e^{2t}]^3}\left\{\alpha^{(2)}_{1l}+\frac{6\alpha^{(2)}_{0l}\alpha^{(1)}_1}{[\alpha^{(1)}_0+e^{2t}]}+36\alpha^{(2)}_{0l}\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{-\infty}^{\infty}\frac{\alpha^{(2)}_{0m}\exp(-2\sigma_m s_2+2s_2)}{[\alpha^{(1)}_0+e^{2s_1}]^2[\alpha^{(1)}_0+e^{2s_2}]^2}\right.$$
$$\times\,dk_m\,ds_2\,ds_1-\mathscr{P}_{ll'}\left[3\int_{t_0}^{t}\int_{-\infty}^{\infty}\frac{\alpha^{(3)}_{0m(l-m)}\exp(2s+\sigma_{l,m(l-m)}s)}{[\alpha^{(1)}_0+e^{2s}]^2}\,dk_m\,ds\right.$$
$$-18\int_{t_0}^{t}ds_1\,\frac{\exp(\sigma_{l,m(l-m)}s_1+2s_1)}{[\alpha^{(1)}_0+e^{2s_1}]^2}\,\mathscr{P}_{m(l-m)l'}\int_{t_0}^{s_1}ds_2\int_{-\infty}^{\infty}dk_m$$
$$\times\,\alpha^{(2)}_{0l'}\,\frac{\alpha^{(2)}_{0(l-m)}\exp(\sigma_{m,l'(l-m)}s_2+2s_2)}{[\alpha^{(1)}_0+e^{2s_2}]^2}$$
$$\left.\left.+3\alpha^{(2)}_{0l}\int_{t_0}^{t}\int_{-\infty}^{\infty}\alpha^{(2)}_{0m}\,\frac{\exp(-2\sigma_m s+2s)}{[\alpha^{(1)}_0+e^{2s}]^3}\,dk_m\,ds\right]\right\}. \tag{2.61}$$

Higher-order terms in the outer expansions for the cumulants $\tilde{Q}^{(n)}$ can be determined in a similar manner. The set of closure equations (2.48)–(2.51) for the leading terms in the outer expansions contains all the terms in the corresponding set for the inner expansions plus certain of the non-linear terms. It turns out that the outer solution is valid all the way back to $t = 0$. The matching then is especially easy, as we can match the unknown constants in the outer solution directly to the initial conditions. The arbitrary functions in (2.52)–(2.61) are given by

$$\left.\begin{aligned}\alpha^{(1)}_0 &= \frac{1}{\epsilon^2}-1, \quad \alpha^{(2)}_{0l} = \frac{H^{(2)}_l}{\epsilon^4\beta}, \ldots,\\ \alpha^{(n)}_{0ll'\ldots l^{(n-2)}} &= \frac{H^{(n)}_{ll'\ldots l^{(n-2)}}}{\epsilon^{2n}\beta^{n-1}} \quad (n \geqslant 2), \quad \alpha^{(n)}_1 = 0 \quad (n \geqslant 1) \quad \text{if} \quad t_0 = 0.\end{aligned}\right\} \tag{2.62}$$

Thus, we can write the outer expansions for the Fourier space cumulants as

$$\tilde{Q}^{(1)} \sim \frac{\epsilon e^t}{[1+\epsilon^2(e^{2t}-1)]^{\frac{1}{2}}}$$
$$\times\left\{1-\frac{3\epsilon^2}{[1+\epsilon^2(e^{2t}-1)]}\int_0^t\int_{-\infty}^{\infty}H^{(2)}_m\frac{\exp(-2\sigma_m s+2s)}{[1+\epsilon^2(e^{2t}-1)]^2}\,dk_m\,ds+\ldots\right\}, \tag{2.63}$$

$$\tilde{Q}_l^{(2)} \sim \frac{\epsilon^2 \exp(2t-2\sigma_l t)}{\beta[1+\epsilon^2(e^{2t}-1)]^3}\left\{H_l^{(2)} + 36\epsilon^4 H_l^{(2)} \int_0^t \int_0^{s_1} \int_{-\infty}^{\infty} \frac{H_m^{(2)} \exp(-2\sigma_m s_2 + 2s_2)}{[1+\epsilon^2(e^{2s_1}-1)]^2 [1+\epsilon^2(e^{2s_2}-1)]^2}\right.$$

$$\times dk_m\, ds_2\, ds_1 - \mathscr{P}_{ll'}\left[\epsilon^2 \int_0^t \int_{-\infty}^{\infty} \frac{3H^{(3)}_{m(l-m)} \exp(2s+\sigma_{l,m(l-m)}s)}{[1+\epsilon^2(e^{2s}-1)]^2}\, dk_m\, ds\right.$$

$$-18\epsilon^4 \int_0^t ds_1 \frac{\exp(\sigma_{l,m(l-m)}s_1 + 2s_1)}{[1+\epsilon^2(e^{2s}-1)]^2}\, \mathscr{P}_{m(l-m)l'} \int_0^{s_1} ds_2$$

$$\times \int_{-\infty}^{\infty} dk_m \frac{H_{l'}^{(2)} H_{l-m}^{(2)} \exp(\sigma_{m,l'(l-m)}s_2 + 2s_2)}{[1+\epsilon^2(e^{2s_2}-1)]^2} + 3\epsilon^2 H_l^{(2)}$$

$$\left.\left.\times \int_0^t \int_{-\infty}^{\infty} \frac{H_m^{(2)} \exp(-2\sigma_m s + 2s)}{[1+\epsilon^2(e^{2s}-1)]^3}\, dk_m\, ds\right] + \ldots\right\}, \tag{2.64}$$

$\vdots$

$$\tilde{Q}^{(n)}_{ll'\ldots l^{(n-2)}} \sim \frac{\epsilon^n \exp(nt - \sigma_{ll'\ldots l^{(n-1)}}t)}{\beta^{n-1}[1+\epsilon^2(e^{2t}-1)]^{(3n/2)}} \{H^{(n)}_{ll'\ldots l^{(n-2)}} + \ldots\}$$

$$(n \geqslant 2, \quad k_l + k_{l'} + \ldots + k_{l^{(n-1)}} = 0). \tag{2.65}$$

Several comments are in order regarding (2.63)–(2.65). First, for $\epsilon^2 e^{2t} \ll 1$, the terms involving $1+\epsilon^2(e^{2t}-1)$ can be expanded in binomial series. By this procedure, one can clearly recover the corresponding inner expansions. When $\epsilon^2 e^{2t} \geqslant O(1)$, such binomial series expansions are no longer valid (or useful), which explains why the inner expansions become disordered on this time scale. The above expressions were constructed on the basis of an expansion in powers of $\beta(\epsilon) = 1/(\log 1/|\epsilon|)^{\frac{1}{2}}$; but β does not appear explicitly in (2.63)–(2.65). This seeming contradiction is removed when one examines the long time behaviour (for $\epsilon^2 e^{2t} \geqslant O(1)$) of the time integrals in these equations. This we shall do below. Fortunately, it turns out that the outer expansions remain uniformly valid for all $t \to +\infty$. Thus the long term behaviour of the cumulants is governed by the leading terms in each of the outer expansions. For $\epsilon^2 e^{2t} \gg 1$ we have that

$$Q^{(1)} \sim 1, \tag{2.66}$$

$$Q^{(n)}_{ll'\ldots l^{(n-2)}} \sim \frac{\exp(-2nt - \sigma_{ll'\ldots l^{(n-1)}}t)}{\epsilon^{2n}} H^{(n)}_{ll'\ldots l^{(n-2)}} \quad (n \geqslant 2), \tag{2.67}$$

which, in terms of the nth order correlations, implies that

$$\langle u(x,t)\rangle \sim 1, \tag{2.68}$$

$$\langle u(x,t)\, u(x+r,t)\ldots u(x+r^{(n-2)},t)\rangle \sim 1 \quad (n \geqslant 2). \tag{2.69}$$

The interpretation of these results is that the long time behaviour of our system is one of perfect correlation.

We shall conclude this section with an investigation of some of the integrals appearing in (2.63)–(2.65) in order to support our claim that for $t \geqslant 0(1/\beta^2)$, the appropriate expansions parameter for the outer expansions is $\beta(\epsilon)$. First we shall examine the following term in (2.63):

$$\epsilon^2 \int_0^t \int_{-\infty}^{\infty} \frac{H_m^{(2)} \exp(-2\sigma_m s + 2s)}{[1+\epsilon^2(e^{2s}-1)]^2}\, dk_m\, ds. \tag{2.70}$$

We shall show that for $\epsilon^2 e^{2t} \geqslant O(1)$, this expression is $O(\beta)$ as it should be. As is well known $H_m^{(2)}$ must be a non-negative function of k_m, and $H^{(2)}(0) = \max H^{(2)}(k_m)$. Further, we have required that it be an ordinary function such that

$$\int_{-\infty}^{\infty} H_m^{(2)}\, dk_m < \infty.$$

Thus,

$$\epsilon^2 \int_0^t \int_{-\infty}^{\infty} \frac{H_m^{(2)} \exp(-2k_m^2 s + 2s)}{[1+\epsilon^2(e^{2s}-1)]^2}\, dk_m\, ds \leqslant \epsilon^2 H^{(2)}(0) \int_0^t \int_{-\infty}^{\infty} \frac{\exp(-2k_m^2 s + 2s)}{[1+\epsilon^2(e^{2s}-1)]^2}\, dk_m\, ds. \tag{2.71}$$

It can easily be shown that the orders of magnitude, with respect to ϵ, of the two terms in (2.71) are the same. Therefore, we shall work with the second expression:

$$\epsilon^2 H^{(2)}(0) \int_0^t \int_{-\infty}^{\infty} \frac{\exp(-2k_m^2 s + 2s)}{[1+\epsilon^2(e^{2s}-1)]^2}\, dk_m\, ds = \frac{\epsilon^2\sqrt{\pi}}{\sqrt{2}} H^{(2)}(0) \int_0^t \frac{e^{2s}\, ds}{s^{\frac{1}{2}}[1+\epsilon^2(e^{2s}-1)]^2}. \tag{2.72}$$

In this last integral set $y = \epsilon^2(e^{2s}-1)$, $dy = 2\epsilon^2 e^{2s}\, ds$. It becomes

$$\begin{aligned} &\frac{\sqrt{\pi} H^{(2)}(0)}{2} \int_0^{\epsilon^2(e^{2t}-1)} \frac{dy}{\left[\log \dfrac{y+\epsilon^2}{\epsilon^2}\right]^{\frac{1}{2}} [1+y]^2} \\ &\qquad = \frac{\sqrt{\pi} H^{(2)}(0)}{2\sqrt{\log(1/\epsilon^2)}} \int_0^{\epsilon^2(e^{2t}-1)} \frac{dy}{\left[1 - \dfrac{\log(y+\epsilon^2)}{\log \epsilon^2}\right]^{\frac{1}{2}} (1+y)^2} \\ &\qquad = \frac{\sqrt{\pi} H^{(2)}(0)}{\sqrt{2}} \beta \int_0^{\epsilon^2(e^{2t}-1)} \frac{dy}{\left[1 - \dfrac{\log(y+\epsilon^2)}{\log \epsilon^2}\right]^{\frac{1}{2}} (1+y)^2}. \end{aligned} \tag{2.73}$$

As we have the desired parameter β multiplying the last integral, we need only show that this integral is an $O(1)$ quantity for $\epsilon^2 e^{2t} \geqslant O(1)$. This is not difficult, for the integrand can be bounded above and below by functions of y only. For example,

$$\frac{1}{(1+y)^3} \leqslant \frac{1}{\left[1 - \dfrac{\log(y+\epsilon^2)}{\log \epsilon^2}\right]^{\frac{1}{2}} (1+y)^2} \leqslant \frac{1}{y^{\frac{1}{2}}(1+y)^{\frac{3}{2}}} \quad (0 < y < \infty), \tag{2.74}$$

where the integrals of the two bounding functions clearly exist. Therefore, we have the desired result.

The final integral that we shall examine is one in (2.64) which arises from the forcing term in (2.50): namely,

$$\epsilon^4 \int_0^t \int_0^{s_1} \int_{-\infty}^{\infty} \frac{H_l^{(2)} H_{l-m}^{(2)} \exp(\sigma_{m\, l(l-m)} s_2 + \sigma_{l,m(l-m)} s_1 + 2s_1 + 2s_2)}{[1+\epsilon^2(e^{2s_1}-1)]^2 [1+\epsilon^2(e^{2s_2}-1)]^2}\, dk_m\, ds_2\, ds. \tag{2.75}$$

As this integral involves k_l we shall have to transform it back into physical space. For consistency, then, we must have that

$$\epsilon^4 \int_{-\infty}^{\infty} \int_0^t \int_0^{s_1} \int_{-\infty}^{\infty} \frac{H_l^{(2)} H_{l-m}^{(2)} \exp(-2\sigma_l t + ik_l r) \exp(\sigma_{m, l(l-m)} s_2 + \sigma_{l, m(l-m)} s_1 + 2s_1 + 2s_2)}{[1+\epsilon^2(e^{2s_1}-1)]^2 [1+\epsilon^2(e^{2s_2}-1)]^2} dk_m\, ds_2\, ds_1\, dk_l = O\left(\frac{\beta}{t^{\frac{1}{2}}}\right) \quad \text{for} \quad \epsilon^2 e^{2t} \geqslant O(1). \quad (2.76)$$

The $1/t^{\frac{1}{2}}$ is the usual diffusion effect. As before, we equivalently examine (on the basis $|r| \ll t^{\frac{1}{2}}$)

$$\epsilon^4 \int_{-\infty}^{\infty} \int_0^t \int_0^{s_1} \int_{-\infty}^{\infty} \frac{\exp\{[k_m^2 - k_l^2 - (k_l - k_m)^2] s_2 + [k_l^2 - k_m^2 - (k_l - k_m)^2] s_1 - 2k_l^2 t + 2s_1 + 2s_2\}}{[1+\epsilon^2(e^{2s_1}-1)]^2 [1+\epsilon^2(e^{2s_2}-1)]^2} \times dk_m\, ds_2\, ds_1\, dk_l. \quad (2.77)$$

The k_l and k_m integrations can be performed without difficulty. This leaves us with

$$\frac{\epsilon^4 \pi}{2t^{\frac{1}{2}}} \int_0^t \int_0^{s_1} \frac{e^{2(s_1+s_2)}\, ds_2\, ds_1}{s_1^{\frac{1}{2}}\left(1+\dfrac{s_2}{t}\right)^{\frac{1}{2}} \left[1 - \dfrac{(s_1+s_2)^2}{4s_1(s_2+t)}\right]^{\frac{1}{2}} [1+\epsilon^2(e^{2s_1}-1)]^2 [1+\epsilon^2(e^{2s_2}-1)]^2}. \quad (2.78)$$

Before making a change of variables it is convenient to discuss two of the factors appearing in this integral. Because the limits of integration require

$$0 \leqslant s_2 \leqslant s_1 \leqslant t,$$

we have that
$$1 \leqslant 1 + \frac{s_2}{t} \leqslant 2 \quad \text{and} \quad \tfrac{1}{2} \leqslant 1 - \frac{(s_1+s_2)^2}{4s_1(s_2+t)} \leqslant 1. \quad (2.79)$$

Therefore, as these two terms clearly do not affect the order of magnitude of the integral, we shall set them both equal to unity. On this basis (2.78) becomes, after making the change of variables,

$$\left.\begin{array}{c} y = \epsilon^2(e^{2s_1}-1), \quad z = \epsilon^2(e^{2s_2}-1), \\ \dfrac{\pi\beta}{t^{\frac{1}{2}}} \displaystyle\int_0^{\epsilon^2(e^{2t}-1)} dy \int_0^Y \frac{dz}{\left[1 - \dfrac{\log(y+\epsilon^2)}{\log \epsilon^2}\right]^{\frac{1}{2}} (1+y)^2 (1+z)^2}, \end{array}\right\} \quad (2.80)$$

which upon integration with respect to z leaves us with

$$\frac{\pi\beta}{t^{\frac{1}{2}}} \int_0^{\epsilon^2(e^{2t}-1)} \frac{\left[\dfrac{1}{2} - \dfrac{1}{1+y}\right] dy}{\left[1 - \dfrac{\log(y+\epsilon^2)}{\log \epsilon^2}\right]^{\frac{1}{2}} (1+y)^2}. \quad (2.81)$$

We have the desired factor multiplying this integral and by the same argument employed in (2.73) the integral itself is an $O(1)$ quantity for $\epsilon^2 e^{2t} \geqslant O(1)$. Thus the proof is complete. The other integrals appearing in the outer expansions can be handled in a similar fashion.

This concludes the analytical treatment in the present section. It is worth commenting that the diffusive character of the model equation (2.1) is entirely responsible for our being able to close the set of cumulant equations. In other words, we need the β factors appearing in (2.42)–(2.45) in order to obtain an essentially uncoupled set of equations for the leading terms in the outer expansions. The significance of this situation is brought out quite clearly by the study of the statistical initial value problem for the following ordinary differential equation

$$\frac{du}{dt} = u(1-u). \tag{2.82}$$

It turns out that, if one assumes that the initial values of the nth order moments $\langle (u(t))^n \rangle$ are small as in (2.17), the rescaled equations for the outer expansions do not contain any small parameters, as do (2.42)–(2.45). Thus a perturbation approach to finding the long-time behaviour of the system is not possible. We might add that, if one is given the initial probability distribution function for u, it is possible to compute the exact time evolution of the moments because the solution of the integrated equation (2.82) is related in a one-to-one manner to the prescribed initial value.

3. Analysis of equation (1.5)

Having outlined in much detail the asymptotic procedures and the correct scales (§ 2), we now turn to the statistical initial value problem posed by (1.5). We write

$$W(X,t) = u(X,t) + iv(X,t), \quad u, v \text{ real}; \tag{3.1}$$

and the real and imaginary part of (1.5) are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial X^2} &= u(1-u^2-v^2), \\ \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial X^2} &= v(1-u^2-v^2). \end{aligned}\right\} \tag{3.2}$$

The analysis is very similar to § 2 save for the fact that the decay of some of the cumulants for long time is algebraic rather than exponential. Hence, we shall use a more schematic approach so that the reader may follow the essential ideas more closely. We define a more obvious notation:

$$\left.\begin{aligned} \langle u(X,t)\rangle &= (u), \\ \langle v(X,t)\rangle &= (v), \\ \langle u(X,t)\,u(X',t)\rangle &= (uu') + (u)^2, \\ \langle u(X,t)\,v(X',t)\rangle &= (uv') + (u)\,(v), \\ \langle u(X',t)\,v(X,t)\rangle &= (u'v) + (u)\,(v), \\ \langle v(X,t)\,v(X',t)\rangle &= (vv') + (v)^2. \end{aligned}\right\} \tag{3.3}$$

The angle brackets denote the moments and the round brackets denote the corresponding cumulants. By virtue of spatial homogeneity the mean values

(u), (v) are functions only of time; the second-order cumulants (uu'), (uv'), $(u'v)$ and (vv') are functions of the spatial separation $r = X' - X$ and time t, and have the necessary behaviour at large r to permit ordinary Fourier transforms. The prime on the dependent variable denotes its value at X', whereas the unprimed functions take their values at the position X. Writing the hierarchy of equations for these cumulants, we obtain

$$d(u)/dt = (u) - [3(u^2)(u) + (u)^3 + (v^2)(u) + 2(uv)(v) + (u)(v)^2] + \dots, \tag{3.4}$$

$$d(v)/dt = (v) - [3(v^2)(v) + (v)^3 + (u^2)(v) + 2(uv)(u) + (v)(u)^2] + \dots, \tag{3.5}$$

$$\mathscr{L}(uu') = -6(uu')[(u^2) + (u)^2] - 2(uu')[(v^2) + (v)^2] - 2[(u'v) + (uv')][(uv) + (u)(v)] + \dots, \tag{3.6}$$

$$\mathscr{L}(uv') = -4(uv')[(u^2) + (u)^2 + (v^2) + (v)^2] - 2[(uu') + (vv')][(uv) + (u)(v)] + \dots, \tag{3.7}$$

$$\mathscr{L}(u'v) = -4(u'v)[(u^2) + (u)^2 + (v^2) + (v)^2] - 2[(uu') + (vv')][(uv) + (u)(v)] + \dots, \tag{3.8}$$

$$\mathscr{L}(vv') = -6(vv')[(v^2) + (v)^2] - 2(vv')[(u^2) + (u)^2] - 2[(u'v) + (uv')][(uv) + (u)(v)] + \dots, \tag{3.9}$$

where $\mathscr{L}$ is the operator $(\partial/\partial t) - 2(\partial^2/\partial r^2) - 2$.

We have not written down the terms arising from cumulants of order higher than two, since these play no major role in the final outcome (as in § 2). The initial conditions are

$$(u),\ (v) = O(\epsilon), \quad (uu'), (uv'), (u'v), (vv') = O(\epsilon^2). \tag{3.10}$$

The first approximation to the inner expansion can be found by neglecting all the non-linear terms.

The higher terms of the expansion are found by successively iterating the lower order solutions. From § 2 we would expect that the uniformity of this expansion fails after a certain time. As typical examples of the inner expansion we choose (u) and (uu') and write their long-time behaviour in order of magnitude form. By 'long-time' we mean times large compared with unity.

$$(u) \sim \epsilon e^t[O(1) + O(\epsilon^2 e^{2t}) + O(\epsilon^2 e^{2t}/\sqrt{t}) + \dots], \tag{3.11}$$

$$(uu') \sim \frac{\epsilon^2 e^{2t}}{\sqrt{t}}[O(1) + O(\epsilon^2 e^{2t}) + \dots]. \tag{3.12}$$

The $1/\sqrt{t}$ behaviour comes from the usual steepest descents analysis and exhibits that the fastest spectral growth is in the $K = 0$ mode. The above expansions are uniformly valid for times in the range $1 \ll t \ll t_1$, where t_1 is defined by

$$\epsilon^2 e^{2t_1} = 1, \quad t_1 = 1/\beta^2, \quad \text{where} \quad \beta = (\log 1/|\epsilon|)^{\frac{1}{2}}. \tag{3.13}$$

For times comparable to t_1 the uniformity of these expansions fails. However, we again note that the magnitudes to which the cumulants (u), (uu'), $(uu'u'')$, etc. have grown are $1, \beta, \beta^2, \dots$, and thus still form an asymptotically ordered sequence inversely proportional to their order. That is the crucial point in the success of this analysis. We rescale as before,

$$\left.\begin{aligned} (u) = (\widetilde{u}), \quad (v) = (\widetilde{v}),\\ (uu') = \beta(\widetilde{uu'}), \quad (uv') = \beta(\widetilde{uv'}), \quad (u'v) = \beta(\widetilde{u'v}),\\ (vv') = \beta(\widetilde{vv'}), \ldots \end{aligned}\right\} \tag{3.14}$$

The equations for the means now become

$$\frac{d(\widetilde{u})}{dt} = (\widetilde{u}) - (\widetilde{u})^3 - (\widetilde{u})\,(\widetilde{v})^2 - 3\beta(\widetilde{u^2})\,(\widetilde{u}) - \beta(\widetilde{v^2})\,(\widetilde{u}) - 2\beta(\widetilde{uv})\,(\widetilde{v}) + O(\beta^2), \tag{3.15}$$

$$\frac{d(\widetilde{v})}{dt} = (\widetilde{v}) - (\widetilde{v})^3 - (\widetilde{u})^2\,(\widetilde{v}) - 3\beta(\widetilde{v^2})\,(\widetilde{v}) - \beta(\widetilde{u^2})\,(\widetilde{v}) - 2\beta(\widetilde{uv})\,(\widetilde{u}) + O(\beta^2). \tag{3.16}$$

It is convenient to add unity times (3.15) to i ($= \sqrt{-1}$) times (3.16), and set

$$(\widetilde{u}) + i(\widetilde{v}) = \rho e^{i\theta} = \rho\cos\theta + i\rho\sin\theta. \tag{3.17}$$

Equating real and imaginary parts, we obtain

$$d\rho/dt = \rho(1-\rho^2) - \beta\rho(X\cos\theta + Y\sin\theta) + O(\beta^2), \tag{3.18}$$

$$d\theta/dt = -\beta(Y\cos\theta - X\sin\theta) + O(\beta^2), \tag{3.19}$$

where

$$X = 3(\widetilde{u^2})\cos\theta + (\widetilde{v^2})\cos\theta + 2(\widetilde{uv})\sin\theta, \tag{3.20}$$

$$Y = 3(\widetilde{v^2})\sin\theta + (\widetilde{u^2})\sin\theta + 2(\widetilde{uv})\cos\theta. \tag{3.21}$$

The zeroth-order solution is

$$\rho_0^2 = \frac{A\,e^{2t}}{1 + A\,e^{2t}}, \tag{3.22}$$

$$\theta_0 = B, \quad B \text{ a constant}. \tag{3.23}$$

As in § 2, the asymptotic expansions are uniformly valid for all time, and so we can find A and B by matching directly to the initial conditions. There is no loss of generality in taking the phase B to be zero, as this just entails a constant translation of the initial pattern in space. From the initial condition, $A = O(\epsilon^2)$. The equations determining the zeroth-order second-order cumulants are

$$(\mathscr{L} + 6\rho_0^2)\,(\widetilde{uu'})_0 = 0, \tag{3.24}$$

$$(\mathscr{L} + 4\rho_0^2)\,(\widetilde{uv'})_0 = 0, \tag{3.25}$$

$$(\mathscr{L} + 4\rho_0^2)\,(\widetilde{u'v})_0 = 0, \tag{3.26}$$

$$(\mathscr{L} + 2\rho_0^2)\,(\widetilde{uv'})_0 = 0. \tag{3.27}$$

The solutions of (3.26)–(3.27) are

$$(\widetilde{uu'})_0 = \int U(K,0)\,e^{(2-2K^2)t}\left(\frac{1+A}{1+A\,e^{2t}}\right)^3 e^{iKr}\,dK, \tag{3.28}$$

$$(\widetilde{uv'})_0 = \int P(K,0)\,e^{(2-2K^2)t}\left(\frac{1+A}{1+A\,e^{2t}}\right)^2 e^{iKr}\,dK, \tag{3.29}$$

$$(\widetilde{u'v})_0 = \int Q(K,0)\, e^{(2-2K^2)t} \left(\frac{1+A}{1+A\,e^{2t}}\right)^2 e^{iKr}\, dK, \tag{3.30}$$

$$(\widetilde{vv'})_0 = \int V(K,0)\, e^{(2-2K^2)t}\, \frac{1+A}{1+A\,e^{2t}}\, e^{iKr}\, dK, \tag{3.31}$$

where $U(K,0)$, $P(K,0)$, $Q(K,0)$ and $V(K,0)$ are the Fourier transforms of $(\widetilde{uu'})$, $(\widetilde{uv'})$, $(\widetilde{u'v})$ and $(\widetilde{vv'})$ at $t = 0$ and are of order ϵ^2/β. We remark that from spatial homogeneity (uv') and $(u'v)$ are related, $(u(x)\,v(x+r)) = (u(x-r)\,v(x))$, and so only P or Q can be initially prescribed.

At this stage a new feature appears, which was not part of the analysis in § 2. A long-time analysis of (3.31) shows that the decay of the second-order cumulant $(\widetilde{vv'})$ is not exponential but only algebraic:

$$(\widetilde{vv'})_0 \sim V(0,0)\frac{1}{A}\left(\frac{\pi}{2t}\right)^{\frac{1}{2}} \quad \text{as} \quad t\to\infty. \tag{3.32}$$

The behaviour of $(\widetilde{vv'})$ is one of initial growth from order ϵ^2/β to order 1, where it reaches a maximum from which it decays algebraically. The cumulants $(\widetilde{uu'})$, $(\widetilde{uv'})$, $(\widetilde{u'v})$ have a similar early behaviour but decay exponentially. The reason for this difference is that by our selection of $\theta_0 = 0$ we essentially have all the power in the (u) mode. Now it is known from discrete analysis that single rolls, while extremely stable to disturbances in phase, are only marginally stable to rolls $\pi/2$ out of phase. One can see this by testing the stability of the solution $u = 1$, $v = 0$ is the pair of equations,

$$\frac{du}{dt} = u(1-u^2-v^2), \quad \frac{dv}{dt} = v(1-u^2-v^2).$$

Thus, because of the much weakened interaction of (vv') with the mean $(u)^2$, as compared with that of (uu') with the mean $(u)^2$ (compare the terms in (3.24) and (3.27)), the order is achieved much less rapidly than § 2 suggests.

This phenomenon manifests itself again if we look at the first perturbation to the energy (or heat flux) ρ,

$$\rho_1 = \frac{\sqrt{A}\; e^t(1+A)}{(1+A\,e^{2t})^{\frac{3}{2}}}\left[3\int_0^t\int U(K,0)\,\frac{e^{(2-2K^2)\tau}}{(1+A\,e^{2\tau})^2}\,dK\,d\tau + \int V(K,0)\,\frac{e^{(2-2K^2)t}-1}{2-2K^2}\,dK\right]. \tag{3.33}$$

The ratio $\beta\rho_1/\rho_0$ exhibits two behaviours. As discussed in § 2, the first term of this ratio corresponding to the first term on the right-hand side of (3.33) never exceeds $O(\beta)$ and decays exponentially in time. The ratio of β times the second term in (3.33) to ρ_0 is

$$\frac{1}{1+A\,e^{2t}}\int dK\; V(K,0)\,\frac{e^{(2-2K^2)t}-1}{2-2K^2},$$

which never exceeds $O(\beta)$, but it decays algebraically (similar to (vv')) for long time.

An additional feature of interest is the production of a constant phase change in the mean. Using (3.19) and (3.29), we compute the phase (the initial phase was chosen to be zero):

$$\theta = -2\beta \int_0^t \int P(K,0)\, e^{(2-2K^2)\tau} \left(\frac{1+A}{1+A\,e^{2\tau}}\right)^2 dK\, d\tau + O(\beta^2), \tag{3.34}$$

which tends to a constant of order β as $t \to \infty$. Thus the only way in which the final order remembers the initial disorder is in a constant phase jump which depends on the initial cross correlation. This result is not inconsistent with the fact that a single mode grows without change of phase, for in the case of a single mode this cumulant would be zero.

We believe that the phase jump is related to an horizontal advection of temperature due to a definite correlation of the initial fields. The whole system moves to the right or left depending on the sign of this initial coupling. Certainly this is one measurable which could serve as a check on the theory presented, but it would be difficult actually to measure it, because that would require rather detailed initial information.

To sum up: if the initial disturbance field is small, random and has energy spread in the local neighbourhood of the critical wave-number k_c, then the field becomes ordered by selecting the single roll motion corresponding to the wave-number k_c. The initial power spectrum has the heat flux,

$$\langle u^2 + v^2 \rangle = (u)^2 + (v)^2 + \int U(K,0)\, dK + \int V(K,0)\, dK,$$

which shows that initially the whole spectrum in the neighbourhood of k_c is carrying heat and drawing energy from the unstable conduction profile. However, the non-linear couplings are such that even though the second-order cumulants (a measure of the energy in the band) can receive potential energy, their growth is inhibited by coupling with a faster growing mean. After a time $t = 1/\beta^2$ the mean extracts more energy from the band than is being put in from the conduction profile, and eventually leads to decay of the higher order cumulants.

The final state is one in which all the heat is transported across the layer by the mean, which in this description corresponds to the motion with the critical scale k_c. In § 5 we will show that this solution is also the optimal one in the Howard sense.

4. Effect of relative order in the initial conditions

Before we continue to the upper bound question, we wish to test the sensitivity of the approach of centring the fundamental solution about the most critical wave-number. If we make the substitution

$$W(X,T) = W_L(X,T)\, e^{iLX} \tag{4.1}$$

in (1.5), we obtain the equation,

$$\frac{\partial W_L}{\partial T} - 2iL\frac{\partial W_L}{\partial X} - \frac{\partial^2 W_L}{\partial X^2} = (1-L^2)\, W_L - W_L^2 W_L^*, \tag{4.2}$$

which is the solvability condition (in non-dimensional, normalized form) if we centre the neutral solution about $(k_c+L, 0)$. Again making the spatial homogeneity assumption, and taking the average of (4.2), we obtain (again for convenience we use t for T)

$$\frac{d}{dt}\langle W_L\rangle = (1-L^2)\langle W_L\rangle - \langle W_L\rangle^2\langle W_L^*\rangle - \text{higher order cumulants.} \tag{4.3}$$

One might imagine that the arguments we use in § 3 are equally applicable for this case from which one concludes that the mean is driven to a constant, the second-order cumulants decay and the heat flux is $1-L^2$. This would correspond to all the motion in sideband mode L. One could then conclude that the spatial homogeneity assumption (centred around the critical wave-number) was itself the selection mechanism. However, if we interpret the averaging process as spatial averaging this cannot be the case, and we devote the following paragraphs to the resolution of this apparent difficulty. Note that the results would not agree with the previous solution, because, if the W_L field is driven to an ordered field $\langle W_L\rangle = (1-L^2)^{\frac{1}{2}}$, then the mean (average over X) of the W field is zero.

The ensuing analysis exhibits the incredible delicacy by which we achieve the closure of the statistical initial value problem. In addition, we will be enabled to answer the question as to the degree of disorder required for the solution to go through. It will turn out that if, instead of stipulating an initial field which is small, random and having the spectral power equally spread in the neighbourhood of k_c, we ask that the initial field consists of the finite amplitude steady solution corresponding to the discrete sideband mode L plus some noise, then the condition that the noise decays is precisely the Eckhaus stability criterion.

Accordingly we form the hierarchy of equations for the correlation of u_L and v_L, the real and imaginary parts of W_L.

$$\frac{d(u_L)}{dt} = (1-L^2)(u_L) - [3(u_L^2)(u_L) + (u_L)^3 + (u_L)(v_L^2) + 2(u_Lv_L)(v_L) + (u_L)(v_L)^2] + \dots, \tag{4.4}$$

$$\frac{d(v_L)}{dt} = (1-L^2)(v_L) - [3(v_L^2)(v_L) + (v_L)^3 + (u_L^2)(v_L) + 2(u_Lv_L)(u_L) + (u_L)^2(u_L)] + \dots, \tag{4.5}$$

$$\begin{aligned}\mathscr{L}(u_Lu_L') + 2L(\partial/\partial r)[(u_Lv_L') - (u_L'v_L)] = &-6(u_Lu_L')[(u_L^2) + (u_L)^2] \\ &-2(u_Lu_L')[(v_L^2)+(v_L)^2] - 2[(u_L'v_L)+(u_Lv_L')][(u_Lv_L)+(u_L)(v_L)] \\ &-(u_L^3u_L') - (u_Lu_L'v_L^2) - (u_Lu_L'^3) - (u_Lu_L'v_L'^2) - \text{products}\{(u_L^3)(u_L),\dots\},\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathscr{L}(v_Lv_L') + 2L(\partial/\partial r)[(u_Lv_L') - (u_L'v_L)] = &-6(v_Lv_L')[(v_L^2) + (v_L)^2] \\ &-2(v_Lv_L')[(u_L^2)+(u_L)^2] - 2[(u_L'v_L)+(u_Lv_L')][(u_Lv_L)+(u_L)(v_L)] \\ &-(v_L^3v_L') - (v_Lv_L'u_L^2) - (v_Lv_L'^2u_L) - (v_Lv_L'u_L'^2) - \text{products}\{(v_L^3)(v_L),\dots\},\end{aligned} \tag{4.7}$$

$$\begin{aligned}\mathscr{L}(u_Lv_L') - 2L(\partial/\partial r)[(u_Lu_L') + (v_Lv_L')] = &-4(u_Lv_L')[(u_L^2)+(u_L)^2+(v_L^2)+(v_L)^2] \\ &-2[(u_Lu_L')+(v_Lv_L')][(u_Lv_L)+(u_L)(v_L)] \\ &-(u_L^3v_L') - (u_Lv_L^2v_L') - (u_L'^2v_L'u_L) - (u_Lv_L'^3) - \text{products}\{(u_L^3)(v_L),\dots\},\end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathscr{L}(u_L'v_L) + 2L(\partial/\partial r)[(u_Lu_L') + (v_Lv_L')] = &-4(u_L'v_L)[(u_L^2)+(u_L)^2+(v_L^2)+(v_L)^2] \\ &-2[(u_Lu_L')+(v_Lv_L')][(u_Lv_L)+(u_L)(v_L)] \\ &-(u_L'^3v_L) - (u_L'v_L'^2v_L) - (u_L^2v_Lu_L') - (u_L'v_L^3) - \text{products}\{(u_L^3)(v_L),\dots\},\end{aligned} \tag{4.9}$$

where the same notation defined in § 3 has been used, except that now

$$\mathscr{L} = \frac{\partial}{\partial t} - 2\frac{\partial^2}{\partial r^2} - 2(1-L^2).$$

If, as before, the initial values are given,

$$(u_L),\quad (v_L) = O(\epsilon),\quad (u_L u_L'),\quad (u_L v_L'),\quad (u_L' v_L),\quad (v_L v_L') = O(\epsilon^2),$$

then the initial balance is purely a linear one. From (4.6) and (4.7) we see that the initial growth of the mean is

$$(u_L), (v_L) \sim \epsilon\, e^{(1-L^2)t}. \tag{4.10}$$

By introducing the natural combinations $(u_L u_L') + (u_L v_L')$, $(u_L u_L') - (v_L v_L')$, $(u_L v_L') - (u_L' v_L)$ and $(u_L v_L') + (u_L' v_L)$ suggested by the linear parts of (4.6)–(4.9) we obtain

$$\mathscr{L}[(u_L u_L') - (v_L v_L')] = 0, \tag{4.11}$$

$$\mathscr{L}[(u_L v_L') + (u_L' v_L)] = 0, \tag{4.12}$$

$$\mathscr{L}[\{(u_L u_L') + (v_L v_L')\} - i\{(u_L v_L') - (u_L' v_L)\}] - 4iL(\partial/\partial r)\,[\{(u_L u_L') + (v_L v_L')\} - i\{(u_L v_L') - (u_L' v_L)\}] = 0. \tag{4.13}$$

If we define $U(K,0)$, $V(K,0)$, $P(K,0)$ and $Q(K,0)$ to be the initial Fourier transforms of $(u_L u_L') + (v_L v_L')$, $(u_L u_L') - (v_L v_L')$, $(u_L v_L') - (u_L' v_L)$ and $(u_L v_L') + (u_L' v_L)$ respectively, then the solutions of (4.11)–(4.13) and resultant asymptotic behaviour are

$$(u_L u_L') - (v_L v_L') = \int V(K,0)\, e^{iKr}\, e^{(2-2K^2-2L^2)t}\, dK \sim V(0,0)\, e^{2(l-L^2)t}\left(\frac{\pi}{2t}\right)^{\frac{1}{2}}, \tag{4.14}$$

$$(u_L v_L') + (u_L' v_L) = \int Q(K,0)\, e^{iKr}\, e^{(2-2K^2-2L^2)t}\, dK \sim Q(0,0)\, e^{2(1-L^2)t}\left(\frac{\pi}{2t}\right)^{\frac{1}{2}}, \tag{4.15}$$

$$\{(u_L u_L') + (v_L v_L')\} - i\{(u_L v_L') - (u_L' v_L)\} = \int (U(K,0) - iP(K,0))\, e^{iKr}\, e^{2t-2(K+L)^2t}\, dK \sim (U(-L,0) - iP(-L,0))\, e^{-iLr}\, e^{2t}\left(\frac{\pi}{2t}\right)^{\frac{1}{2}}. \tag{4.16}$$

Thus the fastest growing second-order cumulants are associated with the mode $K = -L$ or the most critical wave-number k_c. The essential point is that for $L \neq 0$ the growth of the second-order correlation is faster than the growth of the square of the mean, for after a long time

$$(u_L)^2 = O(\epsilon^2\, e^{2(1-L^2)t}), \tag{4.17}$$

$$(u_L u_L') + (v_L v_L') = O\left(\frac{\epsilon^2\, e^{2t}}{\sqrt{t}}\right). \tag{4.18}$$

Thus, whereas for $L = 0$ the size of the second-order cumulant after the time t_1 at which the inner expansion fails is less than the square of the mean, this is no longer so when L is any finite number as

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}\, e^{-2L^2t}} = \infty.$$

Thus the next balance of the equation hierarchy must include the higher-order cumulants ahead of the products of lower-order ones, and we no longer obtain the immediate and straightforward closure of § 3 (e.g. at times t_1 in the equation for the mean the triple cumulant term which we have neglected to write down is the dominant term). It is clear from (4.15) and (4.16) that the energy all goes back to the most critical mode (the higher cumulants behave similarly), and that this is the natural mode around which to centre the analysis. Though we cannot formally deduce the final state, it is clear, and we may verify that the ordered solution corresponding to $W = 1$ satisfies the equation hierarchy (4.6)–(4.9). $W = 1$ implies $u_L = \cos LX$, $v_L = -\sin LX$ which implies

$$\langle u_L \rangle = \langle v_L \rangle = 0, \tag{4.19}$$

$$(u_L u'_L) + (v_L v'_L) = \cos Lr, \quad (u_L u'_L) - (v_L v'_L) = 0, \tag{4.20}$$

$$(u_L v'_L) - (u'_L v_L) = -\sin Lr, \quad (u_L v'_L) + (u'_L v_L) = 0. \tag{4.21}$$

Note that this is precisely the structure evolving from the initial growth given by (4.14)–(4.16).

$\langle u_L^3 \rangle = 0$, as are all averages of triple products:

$$\begin{gathered}(u_L^3 u'_L) = \langle u_L^3 u'_L \rangle - 3\langle u_L u'_L \rangle \langle u_L^2 \rangle = -\tfrac{3}{8}\cos Lr,\\ (u_L'^3 u_L) = (v_L^3 v'_L) = (v_L'^3 v_L) = -\tfrac{3}{8}\cos Lr,\\ (u_L u'_L v_L^2) = (u'_L u_L v_L'^2) = (v_L v'_L u_L'^2) = (v_L v'_L u_L^2) = -\tfrac{1}{8}\cos Lr,\\ (u_L^3 v'_L) = (u_L v_L'^3) = -(u_L'^3 v_L) = -(u_L v_L'^3) = \tfrac{3}{8}\sin Lr,\\ (u_L v_L^2 v'_L) = (u_L'^2 v'_L u_L) = -(u'_L v_L'^2 v_L) = -(u_L^2 v_L u'_L) = \tfrac{1}{8}\sin Lr.\end{gathered} \tag{4.22}$$

It is readily seen that substitution of the above in (4.4)–(4.9) satisfies these equations.

In order to examine the situation when the initial disturbance field is one of a discrete sideband modal solution immersed in a small random noise field, we test the stability of the solution, $(u_L) = (1-L^2)^{\frac{1}{2}}$, $(v_L) = 0$, all other cumulants zero. The equations for the perturbed averages themselves do yield stability as would be expected, since this is similar to disturbing the solution with fields structurally the same as the disturbance. It is the stability (or instability) of the second-order cumulants that is analogous to perturbations of different structures. The linearized equations for the perturbed second-order cumulants are

$$\left.\begin{aligned}\mathscr{L}(u_L u'_L) + 2L(\partial/\partial r)[(u_L v'_L) - (u'_L v_L)] + 4(1-L^2)(u_L u'_L) &= 0,\\ \mathscr{L}(v_L v'_L) + 2L(\partial/\partial r)[(u_L v'_L) - (u'_L v_L)] &= 0,\\ \mathscr{L}(u_L v'_L) - 2L(\partial/\partial r)[(u_L u'_L) + (v_L v'_L)] + 2(1-L^2)(u_L v'_L) &= 0,\\ \mathscr{L}(u'_L v_L) + 2L(\partial/\partial r)[(u_L u'_L) + (v_L v'_L)] + 2(1-L^2)(u'_L v_L) &= 0,\end{aligned}\right\} \tag{4.23}$$

where
$$\mathscr{L} = \frac{\partial}{\partial t} - 2\frac{\partial^2}{\partial r^2}.$$

The Fourier transforms of these quantities have solutions

$$e^{\lambda_i(K,L)t} \quad (i = 1, 2, 3, 4).$$

For stability we must guarantee that

$$\max_{\substack{K\\1\leqslant i\leqslant 4}} \lambda_i(K,L) < 0,$$

as a steepest descents analysis on the physical space cumulants has its largest contribution from those wave-numbers K which maximize $\lambda(K,L)$. The root that can have positive values is

$$\lambda(K,L) = -2K^2-2+2L^2+2((1-L^2)^2+4K^2L^2)^{\frac{1}{2}}, \tag{4.24}$$

for which the maximum is attained at

$$K^2 = (3L^2-1)(L^2+1)/4L^2, \tag{4.25}$$

whence

$$\lambda(K,\ L) = (3L^2-1)^2/4L^2. \tag{4.26}$$

Thus, in order to attain a positive maximum, it is necessary that $L^2 > \frac{1}{3}$. Otherwise $\lambda_i(K,\ L) < 0$, all i and K and we have stability. The final part of this analysis is precisely the analysis one goes through when dealing with discrete perturbations (Newell & Whitehead 1969).

It is concluded therefore that, if the initial disturbance field is random and small, and does not weight the initial spectrum in favour of a particular wave-number, a selection mechanism is available which chooses the ordered solution (which happens in this case to maximize the heat flux), and which solution (except for a slight phase shift) is otherwise independent from its initial conditions. If, on the other hand, the initial field is partially ordered, then the final field may be one which remembers that that order depends on some stability criterion. We have only looked at the extreme cases, but it is reasonable to suggest that there is a critical amplitude balance between the initial magnitude of the ordered field to that of the noise field which determines whether the final solution is obtained by natural selection or by the initial order. It is of interest to remark that Newell & Whitehead showed that the stability level decreases continuously as the initial amplitude of the ordered field decreases.

5. Optimal solution

We relate the solution obtained from the statistical initial value problem in §3 to the optimal steady solution constrained only by a power integral of the equations

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial X^2} = u(1-u^2-v^2), \tag{5.1}$$

$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial X^2} = v(1-u^2-v^2). \tag{5.2}$$

Multiply (5.1) by u and (5.2) by v, average over X, and obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\langle u^2+v^2\rangle+\left\langle\left(\frac{\partial u}{\partial X}\right)^2+\left(\frac{\partial v}{\partial X}\right)^2\right\rangle = \langle u^2+v^2\rangle-\langle(u^2+v^2)^2\rangle. \tag{5.3}$$

The average $\langle\ \rangle$ is defined by

$$\langle f\rangle = \lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R} f\,dX. \tag{5.4}$$

The second term on the left-hand side in (5.3) results from partial integration. If we look for solutions where averages are time independent, and write

$$\langle (u^2+v^2)^2\rangle = \langle u^2+v^2\rangle^2 + \langle\{(u^2+v^2)-\langle u^2+v^2\rangle\}^2\rangle, \tag{5.5}$$

we obtain,
$$\langle u^2+v^2\rangle^2 - \langle u^2+v^2\rangle + \mu^2(u,v) = 0, \tag{5.6}$$

where
$$\mu^2(u,v) = \left\langle\left(\frac{\partial u}{\partial X}\right)^2+\left(\frac{\partial v}{\partial X}\right)^2\right\rangle + \langle\{(u^2+v^2)-\langle u^2+v^2\rangle\}^2\rangle \geqslant 0. \tag{5.7}$$

We wish to choose the field (u,v), which optimizes the heat flux $\langle u^2+v^2\rangle$, and which is constrained by (5.6). Solving (5.6) and taking the larger root, we obtain

$$\langle u^2+v^2\rangle = \tfrac{1}{2}+\tfrac{1}{2}(1-4\mu^2)^{\frac{1}{2}}.$$

Clearly $\langle u^2+v^2\rangle$ is maximal when $\mu^2 = 0$, which implies

$$\langle u^2+v^2\rangle = 1, \quad u^2+v^2 = 1, \quad u, v \text{ constant}. \tag{5.8}$$

This is precisely the ordered solution obtained in § 3.

6. Discussion

The model equation we have used is a prototype for stability problems, and is useful when the linearized problem correctly predicts the initial instability. A slight modification is required when the instability sets in as a growing oscillation: in this case the relevant equation contains, in addition to diffusion terms, dispersion and group velocity terms. These equations arise as a natural consequence in obtaining a uniform approximation to some generic equation, and are accessible to a statistical initial value approach because they contain *a priori* the fundamental structure of the final solution.

To be specific we will comment on the applicability of (1.5) to the convection problem. The choice of a $W(X,T)$ that satisfies (1.5) ensures that w_0 is a uniform first approximation to the Boussinesq equations for slightly supercritical Rayleigh numbers providing (i) the spectral content of the initial data lies close to the most critical mode $k_c = \pi/2$ and (ii) the motion is two-dimensional. Moreover, the former is not a serious restriction as it is clear that the energy not in the immediate neighbourhood of k_c (and also in the higher harmonics of the vertical eigenfunction structure, e.g. $\sin n\pi z$) decays on the thermal diffusion time scale, t_0. Thus for initial times on the time scale measured in units

$$t_0/((Ra-Ra_c)/Ra_c),$$

(i) is closely satisfied. The three-dimensionality in the problem chooses the cellular *pattern* rather than the *scale*. Essentially there are two decision processes: the first is the choice of the scale of the motion, in which energy, initially distributed in an annular neighbourhood of the critical circle $k_c^2 = \frac{1}{2}\pi$ tends to concentrate on this circle; the second is the choice of the cellular pattern (e.g. single rolls, hexagons (superposition of three rolls)), which choice is dictated by the behaviour of external parameters. The former process is essentially what we have described in this paper; in it the final solution seeks to minimize the effect of the *diffusion* term. The latter process occurs as a result of the non-linear *coupling coefficients* between different modes lying on the critical circle. For example, in the case where the mean temperature profile is a slowly varying function of time, the vertical eigenfunction structure of the neutral solution has a marked asymmetry. As a consequence, the solvability condition (analogous to (1.5)) contains quadratic terms as the vertical eigenfunction can reproduce itself by a quadratic non-linear interaction. As a further consequence, the motion chooses the hexagonal solution as the preferred pattern, presumably because the hexagonal structure can reflect the asymmetry (which a single roll cannot). As a further example, in the case when the parameters are strict constants, the preferred pattern is two-dimensional, that of single rolls. Thus, if the energy is initially distributed continuously along the circle $k^2 = \frac{1}{2}\pi$, we would expect the energy eventually to cluster at discrete points.

As yet, we have not been able to deal with this question from the statistical initial value approach. The reason is that we do not have a convenient description which allows the energy to be continuously distributed on the circle $k^2 = \frac{1}{2}\pi^2$. We saw in §4 how important it is to have the correct fundamental solution in the initial description in order to obtain a straightforward closure. For example, if we were to treat the original Boussinesq equations from the statistical initial value approach, then the moments (spatial averages) would have to reflect the horizontal structure of the vertical and horizontal velocities, which are $\cos k_c x$ and $\sin k_c x$ respectively. As demonstrated by a similar difficulty in §4, this results in all the even moments being non-zero and closure is not as readily obtained.

We also wish to comment further on the fact that the solution to the statistical initial value problem based on (1.5) tends in the limit of long time to the upper bound solution in the Howard sense. It would be unwise to give too much credence to the notion that the average flow field is chosen on the basis of maximal heat flux as the functional $\langle u^2+v^2\rangle$ in our description corresponds to many macroscopic properties of the flow (e.g. dissipation). However, what we have shown is that the upper bound can be reached provided the initial state involves sufficient disorder. These results seem to suggest a means of assessing how close the actual flow fields come to the upper bound fields in more general flow problems. This perhaps could be accomplished by first deducing a set of equations analogous to (1.5), which are equivalent in some asymptotic sense to the Navier–Stokes equations as well as having similar upper bound flow fields, but which are more accessible to a statistical initial value approach. A possible procedure might be to introduce time dependence into the time independent Euler equations obtained from the upper bound analysis. Certainly, as the remarks in §4 brought out, the choice of the correct description of the solution played a major role, not in the final result, but in our ability to obtain an immediate *closure* of the non-linear stochastic problem.

As pointed out earlier, (1.5) was derived on the basis of a slightly supercritical Rayleigh number. Because of technical difficulties most of the past efforts to analyze the stability problem have been restricted to this parameter range. However, experimentally it is found that for a much larger range of Rayleigh number the resultant effect of a small random disturbance, natural to the fluid, is the creation of a steady pattern of rolls. This situation suggests that the statistical initial value approach would be advantageous for that more general problem. Preliminary investigations indicate that this is the case and, indeed, the analysis does not appear to offer any insurmountable difficulties. In the context of the previous paragraph it would be of interest to find a theoretical explanation for the experimentally observed phenomenon of a slight increase in the length scale of the preferred roll as the Rayleigh number is increased (Krishnamurti, private communication).

Finally, we wish to raise a point related directly to our own work. We have been able to show that in most circumstances the preferred scale of the final solution is that of critical; however sufficient order (or lack of smoothness in the initial spectrum) can produce the sideband fields e^{iLX}. Moreover, the reason the sideband field can exist is a consequence of the preservation of its structure by the non-linear term in (1.5). (E.g. $(e^{iLX})^2(e^{iLX})^* \to e^{iLX}$.) But if the initial conditions contain two finite amplitude sideband modes e^{iL_1X}, e^{iL_2X}, then all the spatial harmonics will be generated in time. We ask: does the initial order lead to a state of sufficient *disorder*, so that the statistical selection process has an opportunity to work and reproduce the single roll?

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